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DECOMPOSITION PROGRAMMING AND ECONOMIC PLANNING

by

STEPHEN P. BRADLEY

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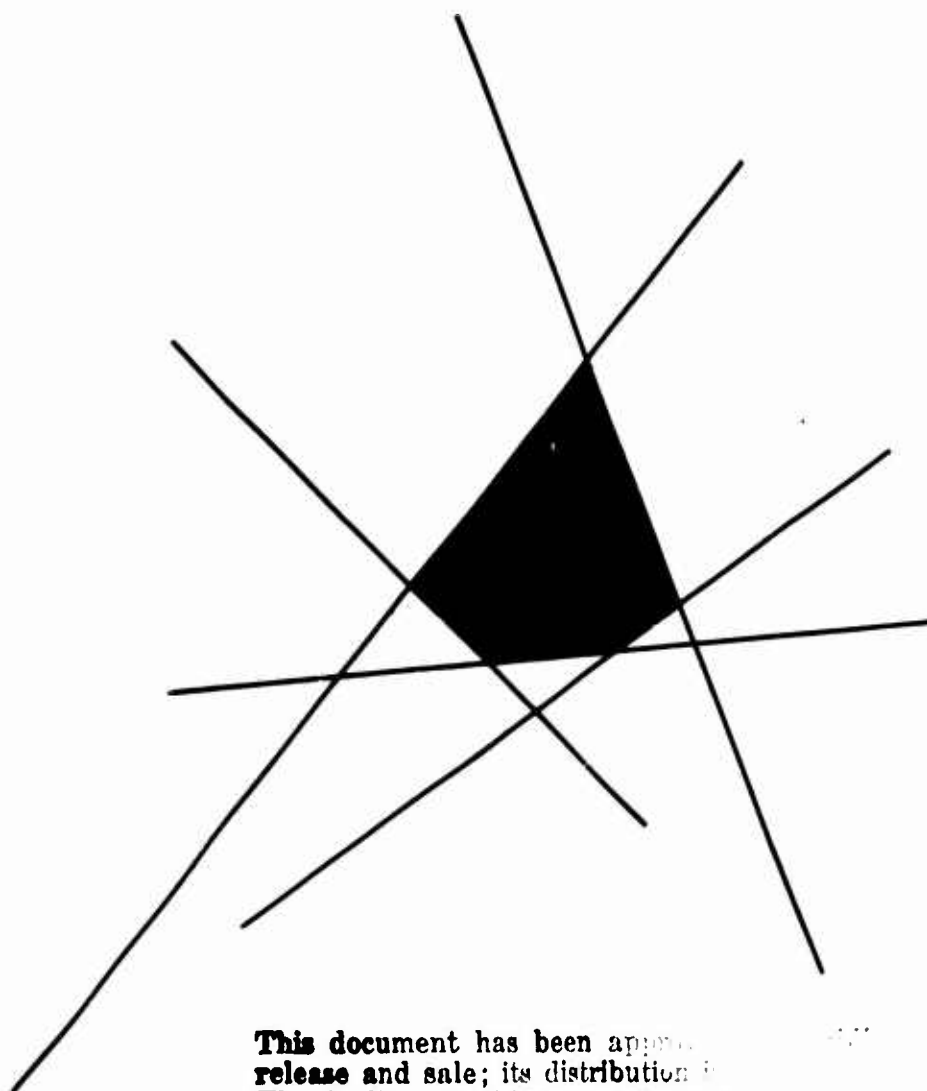
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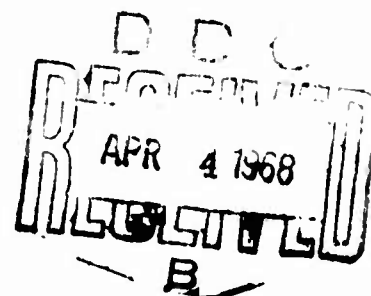
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DECOMPOSITION PROGRAMMING AND ECONOMIC PLANNING

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ABSTRACT

The class of mathematical programming problems whose range is in euclidian n-space but whose domain is an abstract space is considered. A duality theory is presented that relates the constrained maximization problem considered as a function of its right-hand side to the associated Lagrangian maximization problem considered as a function of the Lagrange multipliers. The analysis leads to useful computational procedures.

The constrained maximization problem as a function of its right-hand side is

$$P(y) = \underset{\substack{G(z) \leq y \\ z \, \epsilon \, S}}{\text{Sup}} \; H(z)$$

where S is an arbitrary set, $H(z)$ a functional defined on S, $G(z)$ a vector of functionals defined on S, and y is a vector in euclidian J-space. The associated Lagrangian maximization problem as a function of the Lagrange multipliers

$$L(u) = \underset{z \epsilon S}{\text{Sup}} \; \{H(z) - u \cdot [G(z) - d]\}$$

where u is a vector in euclidian J-space and d is the specific value of y for which the solution of the constrained maximization problem is sought.

Various conditions for a strong duality, in the sense that

$$\underset{y \leq d}{\text{Sup}} \; P(y) = \underset{u \geq 0}{\text{Inf}} \; L(u)$$

are presented. The duality results and some regularity properties of the functions $P(y)$ and $L(u)$ are used as the basis for computational procedures. Dual and primal-dual algorithms for concave programs are given.

Finally, some of the theory presented is applied to the problem of optimal adjustment of the capacity of a firm.

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CHAPTER 1

INTRODUCTION

1.1 RESULTS TO BE PRESENTED

In this paper we consider the class of mathematical programming problems whose range is in euclidian n-space but whose domain of definition is an abstract space. The motivation for including abstract spaces in the analysis is to obtain results for the infinite dimensional spaces of economic interest as well as the usual mathematical programming problems in euclidian n-space. In fact, it was the consideration of an economic problem that gave rise to some of the theory to be presented. The term "decomposition programming" is employed to emphasize the role played in the analysis by the Lagrange multipliers, which historically have been associated with decomposing large-scale programs into subprograms of more reasonable size. Further there is a very close connection, which will be developed in the text, between the computational procedures to be presented and Dantzig's [10] decomposition algorithm for concave programs.

We will extensively investigate the relationship between the constrained maximization problem considered as a function of its right-hand side and its associated Lagrangian maximization problem considered as a function of the Lagrange multipliers. Our analysis attempts always to take that direction that gives insight into new and existing computational procedures. The constrained maximization problem considered as a function of its right-hand side is

$$P(y) = \text{Sup } H(z)$$

$$(1.1-1) \qquad\qquad G(z) \leq y$$

$$z \in S$$

where S is an arbitrary set, H(z) a functional defined on S, G(z)
a vector of functionals defined on S , and y is a vector in euclidian
J-space. When y takes on a specific value, say y = d , then (1.1-1)
is merely the usual constrained maximization problem. The associated
Lagrangian maximization problem considered as a function of its Lagrange
multipliers is

$$(1.1-2) \qquad L(u) = \underset{z \in S}{\text{Sup}} \; \{H(z) - u \, [G(z) - d]\} \quad u \geq 0$$

where u is a vector in euclidian J-space.

When y is specified and S is a set in euclidian n-space, the
conditions under which a solution to (1.1-2) yields a solution to (1.1-1)
are given in the familiar Kuhn-Tucker [19] saddle point theorem. These
conditions have been generalized by Hurwicz [17] to the case were S is
a set in a linear space. However, establishing the optimality conditions
and developing algorithms to satisfy them are two entirely different
problems.

In the development of nonlinear programming algorithms, when S
is a set in euclidian n-space, an extremely important role has been played
by the duality theory of Wolfe [29] and others. This theory assumes that
S is an open set and that the functions are differentiable; then the
"dual" problem becomes

$$\text{Min } \{H(z) - u \cdot G(z)\}$$

(1.1-3)
$$VH(z) - u \cdot VG(z) = 0$$

$$u \geq 0 \qquad z \in S.$$

However, as (1.1-3) is not generally a convex program even when (1.1-1) is appropriately a concave program, the role of the "dual" problem has been a passive one. That is, in the course of most nonlinear programming algorithms "dual" feasible points are constructed which are used to bound the primal objective function and thus act as a termination criteria.

Alternatively, when S is a set in euclidian n-space, a much less emphasized related problem has been considered by Huard [16], Falk [13], and Takahashi [27]. Falk refers to the following as the "auxiliary problem"

(1.1-4)
$$\underset{u \geq 0}{\text{Inf }} L(u)$$

where $L(u)$ is defined in (1.1-2). Huard [16], assumes that $L(u)$ is differentiable and then shows that a solution of (1.1-4) yields a solution of (1.1-1) directly. He also gives an algorithm for solving (1.1-4) in this case. Falk [13], assumes that $H(z)$ is strictly concave, then demonstrates that $L(u)$ must be differentiable, and hence that a solution of (1.1-4) yields a solution of (1.1-1). Takahashi [27], is concerned only with equality constraints but obtains results similar to those of [13]. He further considers linear programming problems and develops a decomposition algorithm that is remarkably similar to that of Balas [3].

For the case where S is an abstract space, computational results are not very plentiful. Everett [12] has developed the generalized Lagrange

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multiplier method to solve (1.1-1), when S is an arbitrary set, using (1.1-2). However, his method is a heuristic one and requires a complete search of the entire Lagrange multiplier space. This dilemma has been partially alleviated by Brocks and Geoffrion [8] as they have shown that Everett's Lagrange multipliers may be generated by linear programming; and in fact, the technique is essentially that of Dantzig's [10] decomposition algorithm for concave programs. However, Dantzig [9] had already shown that this technique could be used for programs defined on an abstract space by applying it to the linear optimal control problem.

In this paper we present a unifying theoretical foundation, and associated computational procedures, for the concepts introduced above. In Chapter 2 we derive many of the properties of the functions $P(y)$ and $L(u)$. $L(u)$ is shown under rather weak assumptions to be a convex continuous function of u for all $u \geq 0$. Then assuming that $P(y)$ is a concave program, the behavior of the resources utilized as a function of their associated Lagrange multipliers is shown to be regular. A conjugate duality theory relating $P(y)$ and $L(u)$ is then presented. The theory is related to the work of Rockafellar [23]; however, the proofs rely on well-known results in the literature. Further, some of the conditions obtained are not demonstrable by the usual separation arguments. The theory is such that any point that is classically "dual" feasible in the sense of Wolfe [29] is also dual feasible in our sense. However, our dual problem is always a convex program, is defined without reference to differentiability, and is meaningful for abstract spaces.

In Chapter 3, we employ the theoretical results of Chapter 2 to develop a dual algorithm. This algorithm is shown to be the logical dual of Dantzig's [10] decomposition algorithm for concave programs. Dantzig's algorithm is then extended to a primal-dual procedure which it is argued should improve the convergence properties of the algorithm. Finally, some nonconvex problems may be handled by the dual algorithm and these are pointed out.

Since an economic problem was the impetus for some of the theoretical results presented, in Chapter 4 we formulate an economic example for which the theory is applicable. The problem, first posed by Arrow, Beckmann and Karlin [1], is the optimal adjustment of the capacity of a firm. We formulate a rather complicated version and then use the notion of a homothetic production function, due to Shephard [24], to reduce it to a more workable form. The example indicates that fairly complex problems may be handled by the algorithms proposed, *if* efficient techniques are available for the solution of the Lagrangian subproblems.

1.2 FORMULATION OF THE CONSTRAINED MAXIMIZATION PROBLEM

We will consider the following rather general formulation of a structured program defined on the product of a finite number of arbitrary sets. For applications these sets will always be subsets of some topological space. The problem specifically is

$$(1.2-1) \qquad \text{Max} \sum_{k \epsilon K} H^k(z^k)$$

$$(1.2-2) \qquad \sum_{k \epsilon K} G_j^k(z^k) \leq d_j \qquad j \epsilon J$$

$$(1.2-3) \qquad z^k \epsilon S^k , k \epsilon K .$$

The index sets $J = \{1,2, \ldots, J\}$ and $K = \{1,2, \ldots, K\}$ are finite. S^k, $k \epsilon K$ are arbitrary sets *not* necessarily in Euclidian n-space. $H^k(z^k)$ and $G_j^k(z^k)$, $j \epsilon J$ are functionals defined on S^k, and $d = (d_1, \ldots, d_J) \epsilon E^J$ is a given vector in Euclidian J-space. Throughout, functionals will *always* be real-valued; but the Sup of a functional may be infinite. Topological spaces will *always* be real linear topological spaces. Finally, if S is said to be a convex set, it will be understood that we mean a convex subset of a linear space. For convenience, we will often use the following notation

$$(1.2-4) \qquad \begin{array}{l} \text{Max } H(z) \\ G(z) \leq d \\ z \epsilon S \end{array}$$

to simplify the writing of (1.2-1), (1.2-2) and (1.2-3). It will be understood that the two systems are identical.

A convenient interpretation of the above formulation is in terms of the so-called cell problem. S^k is then the set of allowable strategies in the

k^{th} cell. $H^k(z^k)$ is the payoff from utilizing strategy $z^k \in S^k$ in the kth cell. $G_j^k(z)$ is the amount of the jth resource consumed by utilizing $z^k \in S^k$ in the kth cell. (If $G_j^k(z^k)$ is negative, it can be thought of as the amount of the jth resource produced.) $\sum_{k \in K} G_j^k(z^k) \leq d_j$ $j \in J$ requires that the amount of resources utilized be bounded above by the available resources. The problem is then to choose that set of strategies $(z^1, \ldots, z^K) \in S^1 \times \ldots \times S^K$ that maximizes the total payoff without exceeding the available resources. The cell problem received its name from considering the associated Lagrangian maximization problem.

$$(1.2\text{-}5) \qquad L(u) = \text{Sup} \left\{ \sum_{k \in K} H^k(z^k) - \sum_{j \in J} u_j \left[\sum_{k \in K} G_j^k(z^k) - d_j \right] \right\}$$
$$z^k \in S^k, \ k \in K$$

By interchanging the order of summation and noting that the problem is defined on the product space, (1.2-5) can be evaluated by solving subproblems over the individual cells of the form

$$(1.2\text{-}6) \qquad L^k(u) = \underset{z^k \in S^k}{\text{Sup}} \ \{H^k(z^k) - \sum_{j \in J} u_j \ G_j^k(z^k)\} \qquad k \in K$$

and noting that

$$(1.2\text{-}7) \qquad L(u) = \sum_{k \in K} L^k(u) + \sum_{j \in J} u_j d_j .$$

Note that if S^k for $k \in K$ are convex polyhedral sets in Euclidian spaces of appropriate dimension, say $S^k \subset E^{n_k}$ for $k \in K$, and $H^k(z^k)$, $G_j^k(z^k)$ for $j \in J$ and $k \in K$ are linear functions, we have the familiar structure of a decomposition linear program. However, the optimization techniques to be presented are basically for convex programming problems as the additional special structure present in linear programming problems is not utilized.

1.3 THE OPTIMALITY CONDITIONS

We will state without proof the well-known results concerning necessary and sufficient optimality conditions for functional programming on general spaces. These conditions are the obvious generalization of the familiar Kuhn-Tucker conditions [19]. The proof may be found in Hurwicz [17] by specializing his theorems V.1 and V.3.1.

Theorem 1·

(i) *Sufficiency*

Let S be an arbitrary set, $H(z)$ a functional defined on S, $G(z)$ a vector of functionals defined on S, and u and d vectors in Euclidian J-space, E^J. If there exists $\hat{z} \in S$ and $\hat{u} \geq 0$ such that

(1.3-1) $H(\hat{z}) - u \cdot [G(\hat{z}) - d] \geq H(\hat{z}) - \hat{u} \cdot [G(\hat{z}) - d] \geq H(z) - \hat{u} \cdot [G(z) - d]$

$\forall~ u \geq 0$ $\forall z \in S$

then $\hat{z}$ is optimal for the constrained maximization problem (1.2-4).

(ii) *Necessity*

Let S be a convex set, $H(z)$ a concave functional defined on S, $G(z)$ a vector of convex functionals defined on S, and u and d vectors in Euclidian J-space, E^J. In order that $\hat{z}$ be optimal for the constrained maximization problem (1.2-4) it is necessary that there exist $\bar{u} \geq 0$ and $u_o \in E^1$, $u_o \geq 0$ such that

(1.3-2) $u_o H(\hat{z}) - u \cdot [G(\hat{z}) - d] \geq u_o H(\hat{z}) - \bar{u} \cdot [G(\hat{z}) - d] \geq u_o H(z) - \bar{u} \cdot [G(z) - d]$

$\forall~ u \geq 0$ $\forall~ z \in S$

Further, if there exists some z^o such that

$$(1.3-3) \qquad G(z^o) < d$$

then $u_o > 0$ and (1.3-2) reduces to (1.3-1) by dividing by u_o; and in this case

$$(1.3-4) \qquad \hat{u} = \frac{\bar{u}}{u_o} .$$

CHAPTER 2

THEORETICAL FOUNDATIONS

2.1 THE LAGRANGIAN MAXIMIZATION PROBLEM

The purpose of this chapter is to develop some of the important relationships between the usual constrained maximization problem and the associated Lagrangian maximization problem. The former will be considered as a function of its right hand side and the latter as a function of the Lagrange multipliers. In this section no assumptions concerning continuity or convexity are made. The constrained maximization problem (1.2-4) as a function of its right hand side is

$$P(y) = \text{Sup } H(z)$$

(2.1-1)
$$G(z) \leq y$$

$$z \in S$$

where the domain of definition of the function $P(y)$ is

(2.1-2) $Y = \{y \mid y \geq G(z), z \in S, P(y) > - \infty \}$.

S is an arbitrary set, $H(z)$ a functional defined on S , and $G(z)$ a J-dimensional vector of functionals defined on S . The associated Lagrangian maximization problem, with $y = 0$, is

(2.1-3) $L_o(u) = \underset{z \in S}{\text{Sup}} \{H(z) - u \cdot G(z)\}$ $u \geq 0$

where $u \in E^J$ is a vector in Euclidian J-space. Throughout, functionals will always assume real values.

The following theorem is a minor extension of the "Main Theorem" of Everett [12] and relates the solution of the Lagrangian maximization problem to a particular constrained maximization problem.

Theorem 2:

Let S be an arbitrary set, $H(z)$ a functional defined on S, and $G(z)$ a vector of functionals defined on S. If $\bar{z}(\bar{u})$ solves the Lagrangian maximization problem for some $\bar{u} \geq 0$, i.e.,

$$(2.1\text{-}4) \quad H(\bar{z}(\bar{u})) - \bar{u} \cdot G(\bar{z}(\bar{u})) = \underset{z \varepsilon S}{\text{Max}} \left\{ H(z) - \bar{u} \cdot G(z) \right\}$$

then $\bar{z}(\bar{u})$ is optimal for the following constrained maximization problem

$$(2.1\text{-}5) \quad \begin{aligned} &\text{Max } H(z) \\ &\quad G(z) \leq \bar{y} \\ &\quad\quad z \varepsilon S \end{aligned}$$

where

$$(2.1\text{-}6) \quad \begin{aligned} \bar{y}_j &= G_j(\bar{z}(\bar{u})) \quad \text{if} \quad \bar{u}_j > 0 \\ \bar{y}_j &\geq G_j(\bar{z}(\bar{u})) \quad \text{if} \quad \bar{u}_j = 0 \end{aligned}$$

Proof:

Since $\bar{z}(\bar{u})$ maximizes the Lagrangrangian for $\bar{u} \geq 0$ by (2.1-4), we have

$$H(\bar{z}(\bar{u})) - \bar{u} \cdot [G(\bar{z}(\bar{u})) - \bar{y}] \geq H(z) - \bar{u} \cdot [G(z) - \bar{y}] \quad \forall z \varepsilon S .$$

Further, since $z(\bar{u})$ is feasible for $G(z) \leq \bar{y}$ by (2.1-6), we have for $u \geq 0$.

$$H(\bar{z}(\bar{u})) - u \cdot [G(\bar{z}(\bar{u})) - \bar{y}] \geq H(\bar{z}(\bar{u})) - \bar{u} \cdot [G(\bar{z}(\bar{u})) - \bar{y}] \quad \forall \ u \geq 0 .$$

Hence we have satisfied the sufficient conditions of Theorem 1, Section (1.3), and thus $\bar{z}(\bar{u})$ solves the constrained maximization problem.

Using Theorem 2, we can now show that solving the Lagrangian maximization problem defines a linear supporting function of the constrained maximization problem considered as a function of its right hand side, i.e., $P(y)$.

Definitions:

(i) $\pi_{\bar{x}}(x) = w \cdot x + \pi_{\bar{x}}(0)$ is a linear supporting function of $F(x)$ at some $\bar{x} \in \Gamma$, where Γ is the domain of definition of $F(x)$, if

(2.1-7) $\qquad \pi_{\bar{x}}(\bar{x}) = F(\bar{x})$ and

(2.1-8) $\qquad \pi_{\bar{x}}(x) \geq F(x) \ \forall \ x \in \Gamma$

(ii) $\pi_{\bar{x}}(x) = w \cdot x + \pi_{\bar{x}}(0)$ is a linear bounding function of $F(x)$, where Γ is the domain of definition of $F(x)$, if

(2.1-9) $\qquad \pi_{\bar{x}}(x) \geq F(x) \ \forall \ x \in \Gamma$

Theorem 3:

Let S be an arbitrary set, $H(z)$ a functional defined on S, and $G(z)$ a vector of functionals defined on S. If $\bar{z}(\bar{u})$ solves the Lagrangian maximization problem for some $\bar{u} \geq 0$, i.e.,

$$H(\bar{z}(\bar{u})) - \bar{u} \cdot G(\bar{z}(\bar{u})) = \underset{z \in S}{\text{Max}}\{H(z) - \bar{u} \cdot G(z)\} \ ,$$

then

$$\pi_{\bar{y}}(y) = \bar{u} \cdot y + L_o(\bar{u})$$

is a linear supporting function of $P(y)$ at $\bar{y} \in Y$ where

$$\bar{y}_j = G_j(\bar{z}(\bar{u})) \quad \text{if} \quad \bar{u}_j > 0$$

$$\bar{y}_j \geq G_j(\bar{z}(\bar{u})) \quad \text{if} \quad \bar{u}_j = 0$$

Proof:

$$\underset{z \epsilon S}{\text{Sup}}\{H(z) - u \cdot G(z)\} = L_o(\bar{u})$$

or equivalently

$$H(z) \leq \bar{u} \cdot G(z) + L_o(\bar{u}) \quad \forall \ z \ \epsilon \ S \ .$$

Consider $\bar{z}(\bar{u})$ defined above such that

$$H(\bar{z}(\bar{u})) = \bar{u} \cdot G(\bar{z}(\bar{u})) + L_o(\bar{u})$$

$$P(\bar{y}) = \underset{\substack{G(z) \leq \bar{y} \\ z \ \epsilon \ S}}{\text{Sup}} H(z) = H(\bar{z}(\bar{u})) = \bar{u} \cdot \bar{y} + L_o(\bar{u}) \ .$$

The above holds with equality by Theorem 2. Hence, we have that

$$P(\bar{y}) = \bar{u} \cdot \bar{y} + L_o(\bar{u})$$

and we now must show that

$$P(y) \leq \bar{u} \cdot y + L_o(\bar{u}) \quad \forall \ y \ \epsilon \ Y \ .$$

Consider arbitrary $\hat{y} \ \epsilon \ Y$. Since

$$H(z) \leq \bar{u} \cdot G(z) + L_o(\bar{u}) \quad \forall \ z \ \epsilon \ S \ ,$$

we can unite

$$P(\hat{y}) = \underset{\substack{G(z) \leq y \\ z \ \epsilon \ S}}{\text{Sup}} H(z) \leq \underset{\substack{G(z) \leq y \\ z \ \epsilon \ S}}{\text{Sup}}\{\bar{u} \cdot G(z) + L_o(\bar{u})\} \leq \bar{u} \cdot \hat{y} + L_o(\bar{u})$$

where the last inequality holds since $\bar{u} \geq 0$. Noting that $\hat{y} \ \epsilon \ Y$ is arbitrary, we have the desired result that

$$P(y) \leq u \cdot y + L_o(\bar{u}) \quad \forall \ y \ \epsilon \ Y .$$

Hence $\pi_{\bar{y}}(y) = \bar{u} \cdot y + L_o(\bar{u})$ is a linear supporting function of $P(y)$ at $\bar{y} \ \epsilon \ Y$.

If the Sup is not attained in the Lagrangian maximization problem, then $\bar{u} \cdot y + L(\bar{u})$ only defines a linear bounding function of $P(y)$, i.e.,

$$(2.1-10) \qquad \bar{u} \cdot y + L(\bar{u}) \geq P(y) \quad \forall \ y \ \epsilon \ Y .$$

We now demonstrate $L_o(u)$ is a convex and continuous function of u for all $u \geq 0$.

Definition:

A function $F(x)$, defined on a convex set Γ , is convex if for $x^1, x^2 \ \epsilon \ \Gamma$ and $0 \leq \alpha \leq 1$

$$(2.1-11) \quad F(\alpha \ x^1 + (1-\alpha)x^2) \leq \alpha \ F(x^1) + (1-\alpha) \ F(x^2) .$$

Theorem 4:

Let S be an arbitrary set, $H(z)$ a functional defined on S , and $G(z)$ a vector of functionals defined on S . Then $L_o(u)$ is convex in u for all $u \geq 0$.

Proof:

To show $L(u)$ convex, let $u^1, u^2 \ \epsilon \ \{u \ | \ u \geq 0\}$ and $0 \leq \alpha \leq 1$

$$L_o(\alpha \ u^1 + (1-\alpha)u^2) = \underset{z \epsilon S}{\text{Sup}} \{H(z) - (\alpha \ u^1 + (1-\alpha)u^2) \cdot G(z)\}$$

$$= \underset{z \epsilon S}{\text{Sup}} \{\alpha[H(z) - u^1 \cdot G(z)] + (1-\alpha)[H(z) - u^2 \cdot G(z)]\}$$

$$\leq \underset{z \varepsilon S}{\text{Sup}}\ \alpha[H(z) - u^1 \cdot G(z)] + \underset{z \varepsilon S}{\text{Sup}}\ (1-\alpha)[H(z) - u^2 \cdot G(z)]$$

$$L_o(\alpha\ u^1 + (1-\alpha)u^2) \leq \alpha\ L_o(u^1) + (1-\alpha)\ L_o(u^2)\ .$$

Hence $L_o(u)$ is convex in u for all $u \geq 0$.

Definitions:

(i)   A functional $F(x)$ on a topological space Γ is lower semi-continuous if the level set

   (2.1-12) $\{x\ |\ F(x) \leq \alpha\}$

   is closed for each real α .

(ii) A functional $F(x)$ on a topological space Γ is upper semi-continuous if the level set

   (2.1-13) $\{x\ |\ F(x) \geq \alpha\}$

   is closed for each real α .

Note that if a functional is both lower and upper semi-continuous it is continuous.

Theorem 5:

Let S be an arbitrary set, $H(z)$ a functional defined on S , and $G(z)$ a vector of functionals defined on S . Then $L_o(u)$ is continuous in u for all $u \geq 0$.

Proof:

By the theorem due to Gale, Klee, and Rockafellar [15], a convex function is upper semi-continuous on a convex polytope. Since the set $\{u \mid u \geq 0\}$, on which $L_o(u)$ is defined, is a convex polytope, we need only show that $L_o(u)$ is lower semi-continuous on this set. Assume that for some real $\bar{\alpha}$ the level set,

$$\{x \mid L_o(u) \leq \bar{\alpha}\}$$

is not closed. Then there exists a sequence $\{u^n\} \to \bar{u}$ such that $u^n \in \{u \mid u \geq 0\}$ $\forall n$ and

$$(*) \qquad L_o(u^n) \leq \bar{\alpha} < L(\bar{u}) \ \forall \ n \ .$$

However,

$$L_o(u^n) = \operatorname*{Sup}_{z \in S}\{H(z) - u^n \cdot G(z)\} \ ,$$

or equivalently

$$L_o(u^n) \geq H(z) - u^n \cdot G(z) \ \forall \ z \in S \ .$$

Since $\{u^n\} \to \bar{u}$, for n sufficiently large, say $n > N$, we have

$$L_o(u^n) \geq H(z) - \bar{u} \cdot G(z) \ \forall \ z \in S, \ n > N$$

and hence

$$L_o(u^n) \geq \operatorname*{Sup}_{z \in S}\{H(z) - \bar{u} \cdot G(z)\} = L(\bar{u}) \ \forall \ n > N$$

which is a contradiction of $(*)$. Hence, $L_o(u)$ is lower semi-continuous in u for all $u \geq 0$, which is the desired result.

2.2 CONTINUITY OF THE RESOURCE UTILIZATION FUNCTIONS

One difficulty in employing Lagrangian optimization techniques is the discontinuous behavior of the resources utilized as a function of their associated Lagrange multipliers. However, in the case of the convex program these discontinuities can be completely described.

Definition:

The amount of the jth resource utilized as a function of the Lagrange multipliers is $G_j(\bar{z}(u))$ where the vector $G(\bar{z}(u))$ of such functions is given by

$$(2.2-1) \qquad H(\bar{z}(u)) - u \cdot G(\bar{z}(u)) = \underset{z \in S}{\text{Max}} \left\{ H(z) - u \cdot G(z) \right\}.$$

We now introduce convexity assumptions not previously made and prove the well-known result that a concave program is a concave function of its right hand side.

Definition:

A functional $F(x)$ defined on a convex set Γ is (strictly) concave if for $x^1, x^2 \in \Gamma$ and $0 \leq \alpha \leq 1$

$$(2.2-2) \qquad F(\alpha x^1 + (1-\alpha) x^2) \underset{(>)}{\geq} \alpha F(x^1) + (1-\alpha) F(x^2) .$$

Theorem 6:

Let S be a convex set and $G(z)$ a vector of convex functionals defined on S. If $H(z)$ is a (strictly) concave functional defined on S, then $P(y)$ is a (strictly) concave function defined on $Y = \{y \mid y \geq G(z), z \in S, P(y) > - \infty \}$.

Proof:

Let $y^1, y^2 \in Y$ and define $y^\alpha = \alpha y^1 + (1-\alpha)y^2$,

where $0 \leq \alpha \leq 1$. Let z^1, z^2 and z^α be optimal solutions to (2.1-1) corresponding to y^1, y^2 and y^α respectively. Since z^1 and z^2 are feasible and $G(z)$ convex, then for $0 \leq \alpha \leq 1$ we have

$$G(\alpha \, z^1 + (1-\alpha)z^2) \leq \alpha \, G(z^1) + (1-\alpha) \, G(z^2) \leq \alpha \, y^1 + (1-\alpha)y^2 \, .$$

Hence $\alpha \, z^1 + (1-\alpha)z^2$ is feasible for $P(\alpha \, y^1 + (1-\alpha)y^2)$, but not necessarily optimal. Thus

$$H(z^\alpha) \geq H(\alpha \, z^1 + (1-\alpha)z^2) \, .$$

If $H(z)$ (strictly) concave then

$$H(z^\alpha) \geq H(\alpha \, z^1 + (1-\alpha)z^2) \geq \alpha \, H(z^1) + (1-\alpha) \, H(z^2)$$
$$(>)$$
Thus $P(\alpha \, y^1 + (1-\alpha)y^2) \geq \alpha \, P(y^1) + (1-\alpha) \, P(y^2) \, .$
$$(>)$$

Using Theorems 3, 5, and 6, we now demonstrate that under the usual convexity assumptions, if one or more of the resource utilization functions are discontinuous at some $\bar{u}$, then we are in fact on a linear segment of the function $P(y)$. Further, if the objective function of the constrained maximization problem is strictly convex, then the resource utilization functions are continuous. Hence, the following two theorems.

Theorem 7:

Let S be a convex set, $H(z)$ a concave functional defined on S, $G(z)$ a vector of convex functionals defined on S, and let the indicated Max exist for $u \in \{u \mid u \geq 0\} \cap N(\bar{u})$, where $N(\bar{u})$ is a neighborhood of $\bar{u}$. If some $G_j(\bar{z}(u))$ is discontinuous at $\bar{u}$, where the vector $G(\bar{z}(u))$ of such functions is defined by

$$H(\bar{z}(u)) - u \cdot G(\bar{z}(u)) = \underset{z \in S}{\text{Max}}\{H(z) - u \cdot G(z)\} \ ,$$

that is, there exists a sequence $\{u^n\} \to \bar{u}$ as $n \to \infty$ such that

$$\underset{n \to \infty}{\text{Lim inf }} G(\bar{z}(u^n)) = \hat{y} \neq \bar{y} = \underset{n \to \infty}{\text{Lim sup }} G(\bar{z}(u^n))$$

then for $0 \leq \alpha \leq 1$,

$$P(\alpha \bar{y} + (1-\alpha)\hat{y}) = \alpha P(\bar{y}) + (1-\alpha) P(\hat{y}).$$

Proof:

$$\pi_{y^n}(y) = u^n \cdot y + L_o(u^n)$$

is a linear supporting function of $P(y)$ at $y^n = G(\bar{z}(u^n))$ by Theorem 3. As $\{u^n\} \to \bar{u}$, $L_o(u^n) \to L_o(\bar{u})$ by Theorem 5, thus

$$\pi_{\hat{y}}(y) = \pi_{\bar{y}}(y) = \bar{u} \cdot y + L_o(\bar{u}) \ .$$

Hence, $\bar{u} \cdot y + L_o(\bar{u})$ is a linear supporting function of $P(y)$ at both $\bar{y}$ and $\hat{y}$, but $\hat{y} \neq \bar{y}$ by the assumed discontinuity of some $G_j(\bar{z}(u))$ at $\bar{u}$. For $0 \leq \alpha \leq 1$,

$$\alpha: \bar{u} \cdot \bar{y} + L_o(\bar{u}) = P(\bar{y})$$
$$(1-\alpha): \bar{u} \cdot \hat{y} + L_o(\bar{u}) = P(\hat{y})$$

Thus, $\bar{u}(\alpha \bar{y} + (1-\alpha)\hat{y}) + L_o(\bar{u}) = \alpha P(\bar{y}) + (1-\alpha) P(\hat{y})$ and by the concavity of $P(y)$, implied by Theorem 6 and the concavity of $H(z)$, we have

$$(*) \qquad \bar{u} \cdot (\alpha \bar{y} + (1-\alpha)\hat{y}) + L_o(\bar{u}) \leq P(\alpha \bar{y} + (1-\alpha)\hat{y}) \ .$$

Now since $\bar{u} \cdot y + L_0(\bar{u})$ is a linear supporting function of $P(y)$

$$\bar{u} \cdot y + L_0(\bar{u}) \geq P(y) \qquad \forall y \epsilon Y .$$

$(\alpha \bar{y} + (1-\alpha)\hat{y}) \epsilon Y$, since $G(z)$ is a vector of convex functions and S is a convex set. Hence

$$\bar{u} \cdot (\alpha \bar{y} + (1-\alpha)\hat{y}) + L_0(\bar{u}) \geq P(\alpha \bar{y} + (1-\alpha)\hat{y})$$

which, with (*), implies

$$\bar{u} \cdot (\alpha \bar{y} + (1-\alpha)\hat{y}) + L_0(\bar{u}) = P(\alpha \bar{y} + (1-\alpha)\hat{y})$$

$$\alpha[\bar{u} \cdot \bar{y} + L_0(\bar{u})] + (1-\alpha)[\bar{u} \cdot \hat{y} + L_0(\bar{u})] = P(\alpha \bar{y} + (1-\alpha)\hat{y})$$

$$\alpha P(\bar{y}) + (1-\alpha) P(\hat{y}) = P(\alpha \bar{y} + (1-\alpha)\hat{y}) .$$

Theorem 8:

Let S be a convex set, $G(z)$ a vector of convex functionals defined on S, and let the indicated Max exist for $u \epsilon \{u \mid u \geq 0\} \cap N(\bar{u})$, where $N(\bar{u})$ is a neighborhood of $\bar{u}$. If $H(z)$ is a *strictly* concave functional defined on S, then $G(\bar{z}(u))$, defined by

$$H(\bar{z}(u)) - u \cdot G(\bar{z}(u)) = \underset{z \epsilon S}{\text{Max}} \{H(z) - u \cdot G(z)\} ,$$

is a vector of functionals continuous at $\bar{u}$.

Proof:

Since $H(z)$ is strictly concave, $H(z)$ is also concave and the hypothesis of Theorem 6 is satisfied. Therefore, if some $G_j(\bar{z}(u))$ is discontinuous at $\bar{u}$, that is, there exists a sequence $\{u^n\} \to \bar{u}$ such that

$$\liminf_{n \to \infty} G(\bar{z}(u^n)) = \hat{y} \neq \bar{y} = \limsup_{n \to \infty} G(\bar{z}(\hat{u}))$$

then for $0 \leq \alpha \leq 1$,

$$P(\alpha \bar{y} + (1-\alpha)\hat{y}) = \alpha P(\bar{y}) + (1-\alpha) P(\hat{y}) .$$

But $H(z)$ is *strictly* concave implies $P(y)$ is *strictly* concave by Theorem 6 and thus for $\bar{y} \neq \hat{y}$

$$P(\alpha \bar{y} + (1-\alpha)) > \alpha P(\bar{y}) + (1-\alpha) P(\hat{y})$$

which is a contradiction. Hence $G(\bar{z}(u))$ is a vector of functions continuous at $\bar{u}$.

2.3 CONJUGATE DUALITY THEOREMS

In this section we join the present trend toward a general symmetric duality theory and present three computationally useful duality theorems, the proofs of which rely on well-known results in the literature. In order to avoid complicating the presentation with existence considerations, we will for the present employ Sup and Inf instead of Max and Min. The constrained maximization problem will be called the primal problem.

$$(2.3\text{-}1) \qquad \begin{array}{c} \text{Sup} \;\; H(z) \\ G(z) \leq d \\ z \in S \end{array}$$

The feasible region of the primal problem is the set

$$(2.3\text{-}2) \qquad Z = \{z \mid z \in S , G(z) \leq d , H(z) > - \infty\} .$$

The associated Lagrangian maximization problem when the right hand side of the constrained maximization problem is explicitly included is

$$(2.3\text{-}3) \qquad L(u) = \underset{z \in S}{\text{Sup}} \; \{H(z) - u \cdot [G(z) - d]\} .$$

We now introduce a related problem first treated by Huard [16] and referred to by Falk [13] as the auxiliary problem. It will here be called the dual problem.

$$(2.3\text{-}4) \qquad \begin{array}{c} \text{Inf} \;\; L(u) \\ u \geq 0 \end{array}$$

The feasible region of the dual problem is the set

$$(2.3\text{-}5) \qquad U = \{u \mid u \geq 0 , L(u) < + \infty\} .$$

To show that (2.3-1) and (2.3-4) can indeed be considered as dual to one another, we demonstrate the usual results concerning weak ordering of the objectives, existence, infeasibility, and optimality. The following theorem

is essentially due to Karamardian [18]; however, here the domain of definition of the functionals is not restricted to Euclidian n-space.

For completeness we adopt the following conventions, which may in fact be argued by contradiction.

(2.3-6)
$$\sup_{z \epsilon \phi} H(z) = - \infty$$

(2.3-7)
$$\inf_{u \epsilon \phi} L(u) = + \infty$$

Theorem 9:

Let S be an arbitrary set, H(z) a functional defined on S , G(z) a vector of functionals defined on S , and let Z and U be defined by (2.3-2) and (2.3-5) respectively. Then

(i) *Weak Duality*

$$\sup_{z \epsilon Z} H(z) \leq \inf_{u \epsilon U} L(u)$$

(ii) *Existence*

If $Z \neq \phi$ and $U \neq \phi$ then both primal and dual have finite optimal solutions, i.e.,

$$- \infty < \sup_{z \epsilon Z} H(z) < + \infty \quad \text{and} \quad + \infty > \inf_{u \epsilon U} L(u) > - \infty$$

(iii) *Infeasibility*

(a) If $U \neq \phi$ and $\inf_{u \epsilon U} L(u) = - \infty$, then $Z = \phi$

(b) If $Z \neq \phi$ and $\sup_{z \epsilon Z} H(z) = + \infty$, then $U = \phi$

(iv) *Optimality*

If $\hat{z} \epsilon Z$, $\hat{u} \epsilon U$, and $H(\hat{z}) = L(\hat{u})$, then $H(\hat{z}) = \sup_{z \epsilon Z} H(z)$

and $L(\hat{u}) = \inf_{u \epsilon U} L(u)$

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<u>Proof</u>:

(i) Consider any $\bar{z} \in Z$ and $\bar{u} \in U$

$$\underset{z \in S}{Sup}\left\{H(z) - \bar{u} \cdot [G(z) - d]\right\} = L(\bar{u})$$

Since the Sup is not necessarily attained at $\bar{z}$, we have

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] \leq L(\bar{u}) .$$

$\bar{z} \in Z$ implies $G(\bar{z}) \leq d$ and $\bar{u} \in U$ implies $\bar{u} \geq 0$.

Therefore $-\bar{u} \cdot [G(\bar{z}) - d] \geq 0$ and hence

$$H(\bar{z}) \leq H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] \leq L(\bar{u})$$

Since $\bar{z} \in Z$ and $\bar{u} \in U$ are arbitrary we have

$$\underset{z \in Z}{Sup\ H(z)} \leq \underset{u \in U}{Inf\ L(u)}$$

(ii) For any $\bar{z} \in Z$ and $\bar{u} \in U$, we have from (i) and the definitions

of Z and U that $-\infty < \underset{z \in S}{Sup\ H(z)} \leq \underset{u \in U}{Inf\ L(u)} \leq L(\bar{u}) < +\infty$

$$+\infty > \underset{u \in U}{Inf\ L(u)} \geq \underset{z \in S}{Sup\ H(z)} \geq H(\bar{z}) > -\infty$$

(iii) Follows immediately from (ii) by contradiction

(iv) Using (i) and the fact that $\hat{z}$ and $\hat{u}$ may not be optimal for the

primal and dual respectively, we have

$$H(\hat{z}) \leq \underset{z \in Z}{Sup\ H(z)} \leq \underset{u \in U}{Inf\ L(u)} \leq L(\hat{u})$$

But since $H(\hat{z}) = L(\hat{u})$, equality must hold throughout which is the

desired result.

In order to obtain the strong duality result that the objective function of the primal is, under the appropriate assumptions, equal to the objective function of the dual, we will employ some of the general minimax theorems due to Sion [26] and Fan [14]. However, first we must prove a minor extension of a lemma due to Karamardian [18]. Here again the domain of definition of the functionals is not restricted to euclidian n-space.

Lemma 10:

Let S be an arbitrary set, $H(z)$ a functional defined on S, and $G(z)$ a vector of functionals defined on S. If K_1 and K_2 are defined by

$$K_1 = \left\{ u, z \mid z \in S, H(z) - u \cdot [G(z) - d] = \inf_{v \geq 0} \{H(z) - v \cdot [G(z) - d]\} \right\}$$

$$K_2 = \{u, z \mid z \in S, u \geq 0, G(z) \leq d, u \cdot [G(z) - d] = 0\},$$

and the set $\{z \mid z \in S, G(z) \leq d\}$ is not empty, then $K_1 = K_2$.

Proof:

(i) Show $K_1 \subset K_2$

Let $(\bar{z}, \bar{u}) \in K_1$ be arbitrary. Hence $\bar{z} \in S, \bar{u} \geq 0$ and

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] = \inf_{v \geq 0} \{H(\bar{z}) - v \cdot [G(\bar{z}) - d]\}$$

which implies

(*) $0 \leq (\bar{u} - v) \cdot [G(\bar{z}) - d] \quad \forall \ v \geq 0$.

Hence $G_j(\bar{z}) \leq d_j \quad \forall \ j \in J$, since if not, setting

$$v_j = \bar{u}_j \quad \text{when} \quad G_j(\bar{z}) \leq d_j \quad \text{and}$$

$$v_j > \bar{u}_j \quad \text{when} \quad G_j(\bar{z}) > d_j ,$$

leads to a contradiction of (*). Finally, $\bar{u} \geq 0$ and $G(\bar{z}) \leq d$ imply $\bar{u} \cdot [G(\bar{z}) - d] \leq 0$, and letting $v = 0$ in (*) we have $\bar{u} \cdot [G(\bar{z}) - d] \geq 0$; hence $\bar{u} \cdot [G(\bar{z}) - d] = 0$.

Therefore $(\bar{z}, \bar{u}) \in K_2$ and $K_1 \subset K_2$.

(ii)   Show $K_2 \subset K_1$

Let $(\bar{z}, \bar{u}) \in K_2$ be arbitrary. Hence $\bar{z} \in S$ and

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z})-d]=H(\bar{z}) \leq H(\bar{z}) - v \cdot [G(\bar{z}) - d] \qquad \forall v \geq 0 .$$

Or equivalently,

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] = \underset{v \geq 0}{\text{Inf}} \; H(\bar{z}) - v \cdot [G(\bar{z}) - d], \text{ since } \bar{u} \geq 0 .$$

Therefore $(\bar{z}, \bar{u}) \in K_1$ and $K_2 \subset K_1$.

(i)   and   (ii)   imply $K_1 = K_2$.

We may now prove the following duality theorems. Euclidian J-space has the usual topology throughout, and S has an unspecified topology in Theorems 11 and 12, while it is merely a convex set in Corollary 13.

Definitions:

(i) A functional $F(x)$, defined on a convex set Γ , is quasi-convex on Γ if for $x^1, x^2 \in \Gamma$ and $0 \leq \alpha \leq 1$

$$(2.3-8) \qquad F(\alpha x^1 + (1 - \alpha)x^2) \leq \text{Max} \; [F(x^1), F(x^2)]$$

(ii) A functional $F(x)$, defined on a convex set Γ , is quasi-concave on Γ if for $x^1, x^2 \in \Gamma$ and $0 \leq \alpha \leq 1$

$$(2.3-9) \qquad F(\alpha x^1 + (1 - \alpha)x^2) \geq \text{Min} \; [F(x^1), F(x^2)] .$$

Theorem 11: *Duality Theorem* (i)

Let S be a convex, compact subset of a topological space and $H(z) - u \cdot [G(z) - d]$ a functional defined on $S \times \{u \in E^J \mid u \geq 0\}$, quasi-concave and upper semi-continuous in z for all $u \geq 0$. Then

$$\underset{\substack{G(z) \leq d \\ z \in S}}{\text{Sup } H(z)} = \underset{u \geq 0}{\text{Inf } L(u)}$$

Proof:

Since $H(z) - u \cdot [G(z) - d]$ is linear and continuous in u for all $z \in S$, it is quasi-convex and lower semi-continuous in u for all $z \in S$. Noting that $H(z) - u \cdot [G(z) - d]$ is quasi-concave and upper semi-continuous on S for all $u \geq 0$ and that S is compact, we have satisfied the conditions of Sion's[†] minimax theorem [26], and we have

(*) $\underset{\substack{z \in S \; u \geq 0}}{\text{Sup Inf}}\{H(z) - u \cdot [G(z) - d]\} = \underset{\substack{u \geq 0 \; z \in S}}{\text{Inf Sup}}\{H(z) - u \cdot [G(z) - d]\}$.

The right-hand side is by definition the dual problem; and applying Lemma 10, the left-hand side reduces to the primal problem. Hence when the primal problem is feasible, we have

$$\underset{\substack{G(z) \leq d \\ z \in S}}{\text{Sup } H(z)} = \underset{u \geq 0}{\text{Inf } L(u)} \quad .$$

If the primal problem is infeasible due to $S = \phi$, then the desired result is immediate from (2.3-6). Otherwise,

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[†]See Lemma A of Appendix A for a statement of Sion's theorem.

if there does not exist a primal feasible solution, then

$$\text{Inf}_{u \geq 0} \{H(z) - u \cdot [G(z)]\} = -\infty \qquad \forall \, z \in S$$

which implies that $\text{Inf}_{u \geq 0} L(u) = -\infty$ by (*); and

noting equation (2.3-6), the proof is complete.

We define the graph of the function $P(y)$ as the set

(2.3-10) $$Y^o = \{y_o, y \mid y_o \leq P(y)\}$$

recalling that

(2.3-11) $$P(y) = \sup_{\substack{G(z) \leq d \\ z \in S}} H(z)$$

and

(2.3-12) $$Y = \{y \mid y \geq G(z), \, z \in S, \, P(y) > -\infty\} \, .$$

Theorem 12: *Duality Theorem* (ii)

Let S be a convex space, $H(z)$ a concave functional defined on S, and $G(z)$ a vector of convex functionals defined on S. If Y^o, the graph of $P(y)$, is closed then

$$\sup_{\substack{G(z) \leq d \\ z \in S}} H(z) = \inf_{u \geq 0} L(u)$$

Proof:

Since the graph of $P(y)$ is closed and convex, it has a support at every boundary point. Hence if the primal problem has a feasible solution it has an optimal solution. By the necessary conditions of Theorem 1, Section (1.3), if $\hat{z} \in S$ is optimal for the primal problem then there exists $\hat{u} \geq 0$ and $u_o \in E^1$, $u_o \geq 0$ such that

$$u_o H(\hat{z}) - u \cdot [G(\hat{z}) - d] \underset{V \, u \geq 0}{\geq} u_o H(\hat{z}) - \hat{u} \cdot [G(\hat{z}) - d] \geq$$

$$\geq u_o H(z) - \hat{u} \cdot [G(z) - d] , \quad V \, z \, \epsilon \, S$$

Equivalently,

$$\underset{u \geq 0}{\text{Inf}} \{u_o H(\hat{z}) - u \cdot [G(\hat{z}) - d]\} = \underset{z \epsilon S}{\text{Sup}} \{u_o H(z) - \hat{u} \cdot [G(z) - d]\}$$

which, noting that $\hat{z}$ and $\hat{u}$ are particular values, implies

$$\underset{z \epsilon S}{\text{Sup}} \underset{u \geq 0}{\text{Inf}} \{u_o H(z) - u \cdot [G(z) - d]\} \geq$$

$$\geq \underset{u \geq 0}{\text{Inf}} \underset{z \epsilon S}{\text{Sup}} \{u_o H(z) - u \cdot [G(z) - d]\} .$$

If $u_o > 0$, we can divide by u_o , and applying Lemma 10 to the left hand side and the definition of $L(u)$ to the right hand side, we have

$$\underset{\substack{G(z) \leq d \\ z \, \epsilon \, S}}{\text{Sup}} H(z) \geq \underset{u \geq 0}{\text{Inf}} L(u) .$$

Noting the weak duality of Theorem 9 (i) equality must hold which is the desired result. If $u_o = 0$, let $u_o = \epsilon$, and by the same argument,

$$\underset{\substack{G(z) \leq d \\ z \, \epsilon \, S}}{\text{Sup}} H(z) = \underset{u \geq 0}{\text{Inf}} L\left(\frac{u}{\epsilon}\right) = \underset{\epsilon v \geq 0}{\text{Inf}} L(v) = \underset{v \geq 0}{\text{Inf}} L(v)$$

Hence, if the primal problem is feasible the theorem holds. If the primal problem is infeasible due to $S = \phi$, then the desired result is immediate from (2.3-6). Otherwise we can take u arbitrarily large in

$$L(u) = \underset{z \epsilon S}{\text{Sup}} \{H(z) - u \cdot [G(z) - d]\}$$

since for any $z \, \epsilon \, S$ an infeasible primal implies $G_j(z) > d_j$ for some $j \, \epsilon \, J$. Hence $\underset{u \geq 0}{\text{Inf}} L(u) = - \infty$ and noting equation (2.3-6), the proof is complete.

In mathematical programming it is customary to define Max and Min over the extended real line; that is, the points $+\infty$ and $-\infty$ are permissable. Furthermore, it is always tacitly assumed that the primal problem (2.3-1) may be written with Sup replaced by Max. However, these conventions are *not* sufficient to replace Inf by Min in the dual problem as Slater's [17] famous counter-example points out. To insure that Sup and Inf may be replaced by Max and Min respectively in the above duality theorems, we must assume a constraint qualification. Noting the previous proof and the necessary conditions of Theorem 1, we get the following expected result.

<u>Corollary 13:</u> *Duality Theorem* (iii)

> Let S be a convex set, $H(z)$ a concave functional defined on S, and $G(z)$ a vector of convex functionals defined on S. If there exists an optimal solution to the primal problem and z^o such that $G(z^o) < d$, then

$$\begin{array}{ll} \text{Max} \quad H(z) & = \text{Min } L(u) \\ \quad G(z) \le d & \quad\quad u \ge 0 \\ \quad z \in S & \end{array}$$

If there exists an optimal solution to the primal problem and a nonempty interior of the constraints, then Sup and Inf may be replaced by Max and Min respectively throughout this section, except in the definition of $L(u)$. If the Sup is always attained, which is the case if S is compact, then Sup may be replaced by Max also in the definition of $L(u)$. For further comments on the existence of $L(u)$, see Section (2.4).

Theorem 11 gives rather weak conditions on the functionals for a strong duality theorem, as will be brought out in Section (3.4) on Nonconvex Considerations. However, the condition of compactness of S is not always satisfied. When S is not compact and the primal is a concave program with closed graph, Theorem 12 says that a similar strong duality theorem holds.

For completeness of the theory, we note the following corollary, which is an immediate consequence of any of the duality theorems and of equations (2.3-6) and (2.3-7).

<u>Corollary 14</u>: *Unboundedness*

Under the assumptions of *Duality Theorem* (i), (ii), or (iii)

(i)   If $U = \phi$ and $Z \neq \phi$ then $\underset{z \in Z}{\text{Sup}}\ H(z) = +\infty$

(ii)   If $Z = \phi$ and $U \neq \phi$ then $\underset{u \in U}{\text{Inf}}\ L(u) = -\infty$

We may characterize the duality theory presented as a conjugate duality by remarking that if the graph of $P(y)$ is closed $L(u)$ defines a linear supporting function of $P(y)$. For emphasis, the conclusions of the duality theorems may be stated

$$\underset{y \leq d}{\text{Sup}}\ P(y) = \underset{u \geq 0}{\text{Inf}}\ L(u)\ .$$

Finally, we should point out the relationship of our duality with that of Wolfe [29]. If S is an open set in Euclidian n-space, $H(z)$ a concave differentiable function defined on S , $G(z)$ a vector of convex differentiable functions defined on S , then a necessary condition for a maximum or a minimum of

(2.3-14) $H(z) - u \cdot [G(z) - d]$

is that the gradient is zero, i.e.,

(2.3-15) $\nabla H(z) - u \cdot \nabla G(z) = 0\ .$

Recalling the dual problem of Wolfe [29],

$$\text{Min } H(z) - u \cdot [G(z) - d] \; ,$$

(2.3-16)
$$\nabla H(z) - u \cdot \nabla G(z) = 0$$

$$u \geq 0 \qquad z \in S$$

we note that whenever $L(u)$ is evaluated, where

(2.3-17)
$$L(u) = \underset{z \in S}{\text{Sup}} \{ H(z) - u \cdot [G(z) - d] \} \qquad u \geq 0 \; ,$$

a feasible solution of (2.3-16) is determined. However, our dual problem

(2.3-17)
$$\underset{u \geq 0}{\text{Inf } L(u)}$$

is always a convex program, is defined without reference to differentiability, and is meaningful for abstract spaces.

2.4 EXISTENCE AND GEOMETRICAL CONSIDERATIONS

Since $L(u) = L_0(u) + u \cdot d$, we know from Theorems 4 and 5, $L(u)$ is convex and continuous in u for all $u \geq 0$. This, coupled with the duality results of the previous section, suggests a computational procedure based on solving the dual in place of the primal problem. However, two questions remain to be considered. When does a solution for the dual yield a solution for the primal, and how is $L(u)$ evaluated when the Sup is not attained? The following theorem partially answers the first question.

Theorem 15:

Let S be a convex set, $H(z)$ a *strictly* concave functional defined on S, and $G(z)$ a vector of convex functionals defined on S. If $\bar{u}$ solves the dual problem (3.2-4) and the graph of $P(y)$ is closed, then $\bar{z}(\bar{u})$ given by

$$H(\bar{z}(\bar{u})) - \bar{u} \cdot [G(\bar{z}(\bar{u})) - d] = \underset{z \in S}{\text{Max}} \{H(z) - \bar{u} \cdot [G(z) - d]\}$$

solves the primal problem (3.2-1).

Proof:

By Theorem 3, $u \cdot y + L_0(u)$ is a linear supporting function of $P(y)$ at $y = G(\bar{z}(u))$. Hence

$$u \cdot G(\bar{z}(u)) + L_0(u) = P(G(\bar{z}(u))) \quad \text{and}$$

(*) $\quad u \cdot y + L_0(u) \geq P(y) \quad \forall y \in Y = \{y \mid y \geq G(z), z \in S, P(y) > -\infty\}$.

Further, since $\bar{u}$ solves the dual problem and the graph of $P(y)$ is closed, we have by Theorem 14 that

$$P(d) = \underset{\substack{G(z) \leq d \\ z \in S}}{\text{Sup}} H(z) = \underset{u \geq 0}{\text{Inf}} L(u) = \bar{u} \cdot d + L_0(\bar{u}).$$

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Now, assuming that $\bar{z}(\bar{u})$ does *not* solve the primal problem, we have

$$G(\bar{z}(\bar{u})) \neq d \ .$$

Hence we may write the following string of inequalities; the first of which follows from the strict concavity of $P(y)$ implied by the strict concavity of $H(z)$ by Theorem 6.

$$P(\alpha\ G(\bar{z}(\bar{u})) + (1 - \alpha)d) > \alpha\ P(G(\bar{z}(\bar{u}))) + (1 - \alpha)\ P(d)$$
$$= \bar{u} \cdot [\alpha\ G(\bar{z}(\bar{u})) + (1 - \alpha)d] + L_o(u)$$
$$\geq P(\alpha\ G(\bar{z}(\bar{u})) + (1 - \alpha)d)$$

The last inequality follows from (*), and exhibits a contradiction. Hence $\bar{z}(\bar{u})$ must solve the primal problem.

If $H(z)$ is only concave and $\bar{u}$ solves the dual problem, $\bar{z}(\bar{u})$ does not necessarily solve the primal problem. However, the regularity properties demonstrated in (2.2) make it possible to construct an optimal solution for the primal. An algorithm for this is given in Section (3.2).

The second question, concerning when the Sup is attained, is in practice rather easily avoided. Evaluating $L(u)$ is itself an optimization problem for which termination conditions to within some preassigned $\epsilon > 0$ are necessary for all but extreme point techniques that terminate in a finite number of steps. The following theorem demonstrates that if we are within ϵ of the true value of $L(u)$ then we are within ϵ of the optimal value for the associated constrained maximization problem.

Theorem 16:

If $\bar{z}$ is within $\epsilon > 0$ of the optimal solution of the Lagrangian maximization problem for some $\bar{u} \geq 0$, i.e.,

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] + \epsilon \geq \underset{z \in S}{Sup}\{H(z) - \bar{u} \cdot [G(z) - d]\} = L(\bar{u})$$

then $\bar{z}$ is within $\epsilon > 0$ of the optimal solution of the associated constrained maximization problem, i.e.,

$$H(\bar{z}) + \epsilon \geq \underset{\substack{G(z) \leq G(\bar{z}) \\ z \in S}}{\mathrm{Sup}} H(z)$$

Proof:[†]

$$H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] + \epsilon \geq \underset{z \in S}{\mathrm{Sup}}\left\{H(z) - \bar{u} \cdot [G(z) - d]\right\}$$

Hence $H(\bar{z}) - \bar{u} \cdot G(\bar{z}) + \epsilon \geq H(z) - \bar{u} \cdot G(z) \qquad \forall z \in S$

or equivalently

(*) $\quad H(\bar{z}) + \epsilon \geq H(z) + \bar{u} \cdot [G(\bar{z}) - G(z)] \qquad \forall z \in S$

Since (*) holds for all $z \in S$, it must hold for any subset of S, in particular

$$\bar{S} = \{z \mid G(z) \leq G(\bar{z}), z \in S\}$$

Noting that $\bar{u} \geq 0$, we have on the set $\bar{S}$

$$\bar{u} \cdot [G(\bar{z}) - G(z)] \geq 0 \qquad \forall z \in \bar{S}$$

and hence $\qquad\qquad H(\bar{z}) + \epsilon \geq H(z) \qquad \forall z \in \bar{S}$

or equivalently

$$H(\bar{z}) + \epsilon \geq \underset{\substack{G(z) \leq G(\bar{z}) \\ z \in S}}{\mathrm{Sup}} H(z)$$

Hence, even though the Sup may not be attained in the definition of $L(u)$, any optimization technique used to evaluate $L(u)$ that comes within

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[†]

This proof is due to Everett [12].

ϵ of the Sup is sufficient for computational purposes.

Finally, if we are to solve the convex dual problem in place of the primal problem, it will be useful to know the gradient of $L(u)$ at $\bar{u}$ if it exists, or at least a subgradient if it does not.

Definition:

A vector w is a subgradient of a function $F(x)$ at $\bar{x}$ in Γ , where Γ is the domain of definition of $F(x)$, if there exists a scalar $\pi_{\underset{x}{-}}(0)$ such that

$$(3.4-1) \qquad \pi_{\underset{x}{-}}(x) = w \cdot x + \pi_{\underset{x}{-}}(0)$$

is a linear supporting function of $F(x)$ at $\bar{x}$.

Theorem 17:

$[d - G(\bar{z}(\bar{u}))]$ is a subgradient of $L(u)$ at $\bar{u}$, where $G(\bar{z}(\bar{u}))$ is defined by

$$H(\bar{z}(\bar{u})) - \bar{u} \cdot [G(\bar{z}(\bar{u})) - d] = \underset{z \in S}{\text{Max}}\{H(z) - \bar{u} \cdot [G(z) - d]\} \; ;$$

and it is assumed that the indicated Max exists for $\bar{u}$.

Proof:

$$L(u) = \underset{z \in S}{\text{Sup}}\{H(z) - u \cdot [G(z) - d]\} = H(\bar{z}(u)) - u \cdot [G(\bar{z}(u)) - d]$$

$$\geq H(z) - u \cdot [G(z) - d] \qquad \forall \; z \; \epsilon \; S$$

Hence, in particular,

$$L(u) \geq H(\bar{z}(\bar{u})) - u \cdot [G(\bar{z}(\bar{u})) - d]$$

or equivalently

$$-L(u) \leq u \cdot [G(\bar{z}(\bar{u})) - d] - H(\bar{z}(\bar{u})) \qquad \forall \; u$$

which implies that there exists $\pi_{\bar{u}}(0)$ such that

$[G(\bar{z}(\bar{u})) - d] \cdot u + \pi_{\bar{u}}(0)$ is a linear supporting function of

$-L(u)$ at $\bar{u}$. Hence $[d - G(\bar{z}(\bar{u}))]$ is a subgradient of

$L(u)$ at $\bar{u}$.

If the indicated Max does not exist for $\bar{u}$, then there does not exist

a support of $P(y)$ with gradient $\bar{u}$. However if $\bar{z}$ comes within ε of

the optimal solution, i.e.,

$$(3.4-2) \qquad H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d] + \varepsilon \geq \underset{z \epsilon S}{\mathrm{Sup}}\{H(z) - \bar{u} \cdot [G(z) - d]\}$$

then $[d - G(\bar{z})]$ can be used as an approximate gradient.

CHAPTER 3

COMPUTATIONAL PROCEDURES

3.1 THE DECOMPOSITION ALGORITHM FOR CONCAVE PROGRAMS

The algorithms based on the theory presented in the previous chapter
are very closely tied to Dantzig's [10] decomposition algorithm for concave
programs, often referred to as generalized programming; and a certain
familiarity with this work will be assumed. However, a brief sketch of the
important concepts is in order as the convergence of this algorithm will be
assumed. In our notation, the problem treated by Dantzig is

$$(3.1-1) \qquad \begin{array}{c} \text{Max} \quad H(z) \\ G(z) \le d \\ z \in S \end{array}$$

where S is a convex compact space, $H(z)$ a concave upper semi-continuous
functional defined on S, and $G(z)$ is a vector of convex lower
semi-continuous functionals defined on S. Actually, Dantzig restricts S
to be a subset of Euclidian n-space and the functionals to be continuous.
However, the first restriction is unnecessary for his proof, as he himself
points out in [9]; and the second may be weakened since a lower semi-continuous
function is bounded below on a compact set.

The technique used to solve (3.1-1) is a generalization of the simplex
method for linear programming in which the coefficients for a column may be
merely points drawn from a convex set. Consider the following generalized
linear program in the variables λ_o and s, which is equivalent to (3.1-1).

$$(3.1-2) \qquad \begin{array}{c} \text{Max} \ y_o \lambda_o \\ y \lambda_o + Is = d \\ \lambda_o = 1 \\ \lambda_o \ge 0, \ s \ge 0 \end{array}$$

where (y_0, y) is a $J + 1$ dimensional vector drawn from the convex set

(3.1-3) $Y^0 = \{(y_0, y) \mid y_0 \leq H(z), y \geq G(z), z \in S\}$.[†]

To initiate the computation, we assume we have a nondegenerate basic feasible solution to (3.1-2), i.e.,

(3.1-4) $\exists \bar{z}_0 \in S$ such that $G(\bar{z}_0) < d$.

Using the Simplex multipliers $\bar{u}$ associated with this basis, the usual pricing out mechanism becomes the following optimization problem,

(3.1-5) $L(\bar{u}) = \underset{z \in S}{\text{Max}} \{H(z) - \bar{u} \cdot [G(z) - d]\} = H(\bar{z}) - \bar{u} \cdot [G(\bar{z}) - d]$.

Hence a new point $[G(\bar{z}), G(\bar{z})] \in Y^0$ is generated and a new basis and associated simplex multipliers, optimal with respect to the points generated thus far, may be calculated; and the procedure continues. The algorithm may be stated as follows.

<u>Algorithm 1:</u> *Decomposition Algorithm for Concave Programs*

   Step (0) Let $R_{(0)} = \{0\}$ be an index set whose only element is associated with $\bar{z}_0$ defined in (3.1-4). Let B be an upper bound on the objective function; and set $B = \infty$ and $n = 0$.

   Step (1) Solve

$$R(u^{(n)}) = \underset{r \in R_{(n)}}{\text{Max}} \sum H(\bar{z}_r) \lambda_r$$

$$\sum_{r \in R_{(n)}} G_j(\bar{z}_r) \lambda_r \leq d_j \qquad j \in J$$

$$\sum_{r \in R_{(n)}} \lambda_r = 1$$

$$\lambda_r \geq 0, r \in R_{(n)}$$

[†]This definition of Y^0 is equivalent to (2.3-10).

yielding an optimal solution $\lambda^{(n)}$ and associated optimal simplex multipliers $u^{(n)}$. Define $z^{(n)} = \sum_{r \in R_{(n)}} \bar{z}_r \lambda_r^{(n)}$

Step (2) If $B - R(u^{(n)}) < \epsilon$, STOP.

Step (3) Solve the Lagrangian maximization problem

$$L_o(u^{(n)}) = \max_{z \in S} \left\{ H(z) - u^{(n)} \cdot [G(z) - d] \right\}$$

yielding $\bar{z}_n$. Set $B = \text{Min} [L(u^{(n)}), B]$ and define the index set $R_{(n+1)}$ by

$$R_{(n+1)} = R_{(n)} \cup \{n\}.$$

Set $n = n + 1$, and go to *Step* (1).

If a nondegenerate basic feasible solution is not initially available, a Phase I procedure that minimizes the degree of infeasibility in the constraints can be used to generate one. In this event, steps (1) and (2) are at first replaced by steps (1') and (2').

Step (1') Solve

$$Q(u^{(n)}) = \text{Max} - \sum_{j \in J} n_j$$

$$\sum_{r \in R_{(n)}} G_j(\bar{z}_r) \lambda_r - n_j \leq d_j \quad j \in J$$

$$\sum_{r \in R_{(n)}} \lambda_r = 1$$

$$\lambda_r \geq 0, \ r \in R_{(n)} \ n_j \geq 0 \ j \in J$$

yielding an optimal solution $(\lambda^{(n)}, n^{(n)})$ and associated optimal simplex multipliers $\pi^{(n)}$. Define

$$z^{(n)} = \sum_{r \in R_{(n)}} \bar{z}_r \lambda_r^{(n)}.$$

Step (2') If $Q(u^{(n)}) = 0$, end Phase I and go to *Step* (1) .

The theorem shown by Dantzig [10] is

Theorem 18:

Let S be a convex compact space, $H(z)$ a concave upper semi-continuous functional defined on S , $G(z)$ a vector of convex lower semi-continuous functionals defined on S , $\hat{z}$ an optimal solution to the primal problem (3.1-1), and let $\{u^{(n)}\}$ be the sequence of multipliers generated by Algorithm 1. If there exists $\bar{z}_o \in S$ such that $G(\bar{z}_o) < d$, then

(3.1-6)
$$\lim_{n \to \infty} u^{(n)} = \hat{u}$$

(3.1-7)
$$\lim_{n \to \infty} H(z^{(n)}) = \mathop{\text{Max}}_{\substack{G(z) \leq d \\ z \in S}} H(z)$$

(3.1-8) $H(\hat{z}) - \hat{u} \cdot [G(\hat{z}) - d] = \mathop{\text{Max}}_{z \in S} \left\{ H(z) - \hat{u} \cdot [G(z) - d] \right\}$

The generalized programming technique is based upon the fact that a convex set can be represented by a convex combination of a sufficiently dense set of the extreme points of the set. Step (3) of the algorithm evaluates $L(u)$ which, by Theorem 3 of Section (2.1) and the compactness of S defines a linear supporting function of

(3.1-9)
$$P(y) = \mathop{\text{Sup}}_{\substack{G(z) \leq y \\ z \in S}} H(z)$$

and hence an extreme point of the set Y^o . The algorithm essentially builds up a convex polyhedral set within the convex set Y^o . At each iteration, the optimal solution over the polyhedral set is found in order to obtain simplex multipliers which are then used to generate another extreme point of Y^o ,

and the procedure continues. Since the optimal value of the primal objective function is bounded above by $L(u)$, being within ε of the minimum value of $L(u)$ generated thus far is used as the termination criteria. The algorithm is referred to as a primal method in our terminology as it works entirely within the convex set Y^o and merely uses the dual as a bound. Further, a primal method is a two phase procedure that first generates a feasible solution and then an optimal solution.

In the statement of the algorithms in this chapter, if S is compact we will replace Sup by Max in the primal problem and in the definition of $L(u)$. However, the constraint qualification is required to replace Inf by Min in the dual.

3.2 A DUAL ALGORITHM

In the previous section, we characterized a primal algorithm as one that depends on the fact a convex set may be represented by a sufficiently dense set of its extreme points. We now consider the logical dual of these techniques and characterize a dual algorithm as one that depends on the fact that a convex set may be represented as the intersection of all supporting hyperplanes. While a primal algorithm works entirely within the convex set Y^o, where

$$(3.2-1) \qquad Y^o = \{y_o, y \mid y_o \leq H(z), y \geq G(z), z \in S\} ,$$

a dual algorithm works entirely outside the convex set Y^o. Recall that the dual problem is

$$(3.2-2) \qquad \underset{u \geq 0}{Inf} \ L(u) ,$$

where the function $L(u)$ is given by

$$(3.2-3) \qquad L(u) = \underset{z \in S}{Sup} \left\{ H(z) - u \cdot [G(z) - d] \right\} .$$

Also, whereas a primal method is a two phase procedure, a dual method moves simultaneously towards feasibility and optimality.

By Theorems 3 and 4 of Section (2.1) we know that $L(u)$ is a convex continuous function of u, for all $u \geq 0$. Further, in Section (2.4), it was shown that whenever $L(u)$ is evaluated at some $\bar{u}$ a subgradient of $L(u)$ at $\bar{u}$ is also determined. Thus we want to minimize a convex continuous function, for which at least a subgradient is readily available at every point, subject only to nonnegativity restrictions on the variables. This would seem to imply that a straight-forward gradient descent technique would be efficient. Huard [16] first and then Falk[13] have proposed this approach; however, in both papers $L(u)$ is essentially assumed differentiable.

The difficulty arises from the fact, pointed out in Section (2.4), that the primal solution $\bar{z}(\hat{u})$ associated with an optimal solution of the dual may not be feasible if $L(u)$ is not differentiable at $\hat{u}$. Therefore, any proposed algorithm that solves the dual problem in order to solve the primal must also construct an optimal *feasible* primal solution.

We first give a gradient descent algorithm for solving the dual problem. Then for those problems where the solution $\bar{z}(\hat{u})$ associated with the optimal dual solution is not primal feasible, a perturbation algorithm is given that constructs a primal feasible solution. The algorithm for the dual is

<u>Algorithm 2:</u> *Gradient Descent Algorithm*

    Step (0) Set $u^{(0)} = +\infty$, $L(u^{(0)}) = +\infty$, $u^{(1)} = 0$, and $n = 1$.

    Step (1) Compute $L(u^{(n)})$ and a . adient of $L(u)$ at $u^{(n)}$ by solving the Lagrangian maximization problem

$$L(u) = \underset{z \in S}{\text{Max}} \left\{ H(z) - u^{(n)} \cdot [G(z) - d] \right\}$$

yielding an optimal solution $\bar{z}(u^{(n)})$.

    Step (2) If $L(u^{(n)}) < L(u^{(n-1)})$, then define

$$\delta_j^{(n)} = \begin{cases} d_j - G_j(\bar{z}(u^{(n)})) & \text{if } G_j(\bar{z}(u^{(n)})) \geq d_j \text{ or } u_j^{(n)} > 0 \\ 0 & \text{if } G_j(\bar{z}(u^{(n)})) < d_j \text{ and } u_j^{(n)} = 0 \end{cases}$$

If $\delta_j^{(n)} = 0$, STOP.

Determine a new set of Lagrange multipliers by

$u^{(n+1)} = u^{(n)} - \theta \delta^{(n)}$ where θ is a scalar such that

$$0 < \theta \leq \text{Min} \left\{ \bar{\theta} , \frac{u_j^{(n)}}{\delta_j^{(n)}} \,\middle|\, \delta_j^{(n)} > 0 \right\}$$

and $\bar{\theta}$ is the maximum step size. Set $n = n + 1$ and go to Step 1.

Step (3) If $0 \leq L(u^{(n)}) - L(u^{(n-1)}) < \epsilon$ and $| u^{(n)} - u^{(n-1)} | < \delta$,

where ϵ and δ are preassigned numbers, then STOP. Otherwise, set $\theta = \theta/2$ and go to *Step* (1).

Theorem 19:

Let S be a convex compact space, $H(z)$ a concave functional defined on S , $G(z)$ a vector of convex functions defined on S , and let $\{u^{(n)}\}$ be the sequence of multipliers generated by Algorithm 2. If there exists $\bar{z} \in S$ such that $G(\bar{z}) \leq d$, then

$$(3.2\text{-}4) \qquad \lim_{n \to \infty} u^{(n)} = \hat{u}$$

$$(3.2\text{-}5) \qquad \lim_{n \to \infty} L(u^{(n)}) = \operatorname*{Inf}_{u \geq 0} L(u)$$

Proof:

$L(u^{(n)})$ is a strictly decreasing sequence of real numbers bounded from below by the primal objective function, which must be finite since $\bar{z}_o$ is primal feasible; hence $L(u^{(n)})$ converges. The sequence $\{u^{(n)}\}$ converges to $\hat{u}$, since termination takes place only if $\delta^{(n)} = 0$ or $| u^{(n)} - u^{(n-1)} | < \delta$, one of which must occur since $L(u)$ is a continuous function of u for all $u \geq 0$ by Theorem 4. Assume, for the purpose of contradiction, that $L(\hat{u})$ is not optimal for the dual. Then let

$$\hat{\delta} = \begin{cases} d_j - G_j(\bar{z}(\hat{u})) & \text{if } G_j(\bar{z}(\hat{u})) \geq d_j \text{ or } \hat{u}_j > 0 \\ 0 & \text{if } G_j(\bar{z}(\hat{u})) < d_j \text{ and } \hat{u}_j = 0 \end{cases}$$
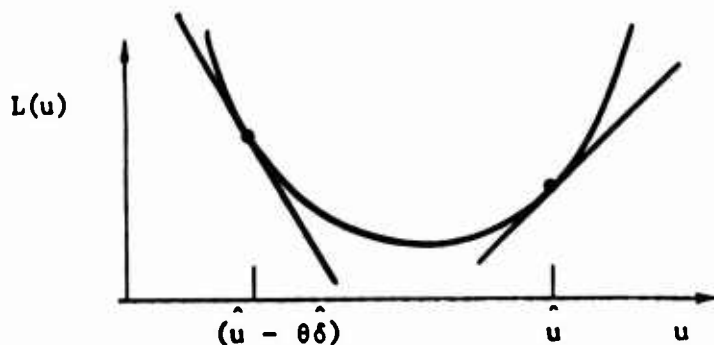
where $\hat{\delta} \neq 0$, since we would then be optimal. Consider $\hat{u} - \theta \hat{\delta}$. By the convexity of $L(u)$ and the termination of the algorithm, we have

(i) $\quad 0 \geq L(\hat{u}) - L(\hat{u} - \theta\hat{\delta}) \geq \theta\hat{\delta} \cdot [D - G(\bar{z}(u - \theta\hat{\delta}))]$;

and by the definition of $\hat{\delta}$, we have

(ii) $\qquad\qquad \theta\hat{\delta}[D - G(\bar{z}(\hat{u}))] > 0$

Hence a subgradient at $\hat{u}$ is positive and a subgradient at $\hat{u} - \theta\hat{\delta}$ is negative. This implies that there exists a point in-between with a subgradient of zero. Hence either $L(\hat{u})$ is optimal to the dual or there exists $\hat{\theta}$ such that $L(\hat{u} - \theta\hat{\delta}) < L(\hat{u})$ and the algorithm would not have terminated, which is a contradiction.

If $\hat{u}$ is an optimal solution to the dual problem such that $\hat{\delta} = 0$, then by the definition of $\hat{\delta}$

(3.2-6) $\qquad$ If $\qquad \hat{u}_j > 0 \quad$ then $\quad G_j(\bar{z}(\hat{u})) = d_j$

$\qquad\qquad$ If $\quad G_j(\bar{z}(\hat{u})) < d_j \quad$ then $\qquad u_j = 0$.

Therefore, $\hat{\delta} = 0$ implies that $\bar{z}(\hat{u})$ is a feasible solution to the primal problem, and hence optimal by Theorem 9(iv) Section (2.3). However, if $\bar{z}(\hat{u})$ is not a feasible solution to the primal, then essentially a Phase I type procedure must be instituted. The following perturbation algorithm will construct the desired optimal feasible solution to the primal.

<u>Algorithm 3:</u>

Step (0) Given $\hat{u}$, the optimal dual solution resulting from Algorithm

2, and $\bar{z}(\hat{u})$ its associated primal infeasible solution, let

$R_{(0)} = \{0\}$ be an index set whose only element is associated

with $\bar{z}(\hat{u})$. Set $u^{(0)} = \hat{u}$ and $n = 0$.

Step (1) Solve

$$Q(u^{(n)}) = \text{Max} - \sum_{j \in J} n_j$$

$$\sum_{r \in R_{(n)}} G_j(\bar{z}_r)\lambda_r - n_j \leq d_j \quad j \in J$$

$$\sum_{r \in R_{(n)}} \lambda_r = 1$$

$$\lambda_r \geq 0 \, , \, r \in R_{(n)}, \, n_j \geq 0, \, j \in J$$

yielding an optimal solution $(\lambda^{(n)}, n^{(n)})$ and associated

optimal simplex multipliers $\pi^{(n)}$. Define $z^{(n)} = \sum_{r \in R_{(n)}} \bar{z}_r \lambda_r^{(n)}$

Step (2) If $Q(u^{(n)}) = 0$, STOP.

Step (3) Otherwise, $Q(u^{(n)}) < 0$ and determine new Lagrange multipliers

by $u^{(n+1)} = u^{(n)} + \theta \pi^{(n)}$ where θ is in the interval

$$0 < \theta \leq \text{Min}\left\{ \bar{\theta}, \frac{u_j^{(n)}}{\pi_j^{(n)}} \, \middle| \, \pi_j^{(n)} < 0 \right\}$$

and $\bar{\theta}$ is an upper bounded on the step size.

Step (4) Solve

$$L(u^{(n)}) = \text{Max}\{H(z) - u^{(n)} \cdot [G(z) - d]\}$$
$$z \in S$$

yielding an optimal solution $\bar{z}_n$. If

$0 \leq L(u^{(n)}) - L(\hat{u}) < \epsilon$, then define the index set $R_{(n+1)}$ by $R_{(n+1)} = R_{(n)} \cup \{n\}$ and go to *Step* (1). Otherwise, set $\theta = \theta/2$ and go to *Step* (4).

Theorem 20:

Let S be a convex compact space, $H(z)$ a concave upper semi-continuous functional defined on S , $G(z)$ a vector of convex lower semi-continuous functionals defined on S , and let $\{u^{(n)}\}$ be the sequence of multipliers generated by Algorithm 3. Then

$$(3.2-7) \qquad \lim_{n \to \infty} Q(u^{(n)}) = 0$$

Proof:

If at each iteration we set $u^{(n)} = \pi^{(n)}$, the algorithm reduces to a Phase I generalized program and such a procedure converges by Theorem 18. However, by Theorem 11 of Section (2.3) and the fact that $\hat{u}$ is an optimal solution for the dual, we have

$$(*) \qquad P(d) = \underset{\substack{G(z) \leq d \\ z \in S}}{\text{Sup}} H(z) = \underset{u \geq 0}{\text{Inf}} L(u) = L(\hat{u}) ,$$

and hence that the primal optimal must occur on a hyperplane whose gradient is $\hat{u}$. Algorithm 3 is merely a Phase I generalized program, restricted to the intersection of the hyperplane defined by (*) and Y_o , and it converges by Theorem 18, Section (3.1) if an extreme point of this set is generated at each iteration. Steps (3) and (4) construct the necessary extreme point in a finite number of steps by the continuity of $L(u)$. Hence the procedure converges and

$$\lim_{n \to \infty} Q(u^{(n)}) = 0 .$$

3.3 A PRIMAL-DUAL ALGORITHM FOR CONCAVE PROGRAMS

One of the computational difficulties with using Dantzig's decomposition algorithm for concave programs is that at times it exhibits poor convergence properties. Poor convergence for any particular problem is due to the shape of the convex set Y^O for that particular problem. Recall

$$(3.3-1) \qquad Y^O = \{y_o, y \mid y_o \le H(z),\ y \ge G(z),\ z \in S\} .$$

The duality theory presented in the previous chapter has the geometrical property that for a particular vector of Lagrange multipliers $\bar{u}$, if the associated primal solution is not unique and therefore on a linear segment, then the associated dual solution is pointed; and vice-versa. Hence, whenever one method exhibits poor convergence properties, the other should not have these difficulties. Therefore it is believed that a primal-dual method will overcome any of the poor convergence properties that either method might exhibit independently.

The following algorithm is a combination of the primal and dual algorithms presented. At each iteration, an extreme point of the set Y^O is generated using the optimal simplex multipliers for that iteration. Then, instead of returning to the linear program as in Algorithm 1, a gradient descent step on $L(u)$ is performed. Hence, a second extreme point of the set Y^O is generated, and finally the procedure returns to the linear program to compute new simplex multipliers.

<u>Algorithm 4</u> *Primal-Dual Algorithm for Concave Programs*

Step (0) Let $R_{(n)} = \{0\}$ be an index set whose only element is associated with $\bar{z}_o$ defined in (3.1-4). Set $B = \infty$ and $n = 0$.

Step (1) Set *flag* = 0 and solve

$$R(u^{(n)}) = \underset{r\epsilon R_{(n)}}{\text{Max}} \sum H(\bar{z}_r)\lambda_r$$

$$\sum_{r\epsilon R_{(n)}} G_j(\bar{z}_r)\lambda_r \leq d_j \quad j \epsilon J$$

$$\sum_{r\epsilon R_{(n)}} \lambda_r = 1$$

$$\lambda_r \geq 0, \ r \epsilon R_{(n)}$$

yielding an optimal solution $\lambda^{(n)}$, and associated simplex

multipliers $u^{(n)}$. Define $z^{(n)} = \sum_{r\epsilon R_{(n)}} \bar{z}_r\bar{\lambda}_r$.

Step (2) If $B - R(u^{(n)}) < \epsilon$, STOP.

Step (3) Solve

$$L(u^{(n)}) = \underset{z\epsilon S}{\text{Max}} \left\{ H(z) - u^{(n)} \cdot [G(z) - d] \right\}$$

yielding $\bar{z}_n$. Set $B = \text{Min} [L(u^{(n)}), B]$ and if $B = L(u^{(n)})$,

set $\bar{u} = u^{(n)}$ and $\bar{z}(\bar{u}) = \bar{z}_n$. Define the index set $R_{(n+1)}$

by $R_{(n+1)} = R_{(n)} \cup \{n\}$. If *flag* = 1 , then go to *Step* (1).

Step (4) Define

$$\delta_j^{(n)} = \begin{cases} d_j - G_j(\bar{z}(\bar{u})) & \text{if } G_j(\bar{z}(\bar{u})) \geq d_j \text{ or } \bar{u}_j > 0 \\ \\ 0 & \text{if } G_j(\bar{z}(\bar{u})) < d_j \text{ and } \bar{u}_j = 0 . \end{cases}$$

Determine a new set of Lagrange multipliers by

$u^{(n+1)} = \bar{u} - \theta\delta^{(n)}$ where θ is a scalar such that

$$0 < \theta \leq \text{Min} \left\{ \bar{\theta} , \frac{\bar{u}_j}{\delta_j^{(n)}} \ \middle| \ \delta_j^{(n)} > 0 \right\}$$

and $\bar{\theta}$ is the maximum step size. Set *flag* = 1, n = n + 1 ,

and go to *Step* (3).

If there does not exist an initial nondegenerate basic feasible solution, as defined in (3.1-4), then a Phase I procedure can be used which is also a primal-dual algorithm. In this event, steps (1) and (2) are at first replaced by steps (1') and (2') of Section (3.1). Alternatively, only dual steps might be performed until a feasible solution is generated. This eliminates the Phase I procedure and has an intuitive appeal since the dual algorithm moves simultaneously towards feasibility and optimality.

Theorem 21:

Let S be a convex compact space, $H(z)$ a concave upper semi-continuous functional defined on S, $G(z)$ a vector of convex lower semi-continuous functionals defined on S, $\hat{z}$ an optimal solution to the primal problem (3.1-1), and let $\{u^{(n)}\}$ be the sequence of multipliers generated by Algorithm 4. If there exists $\bar{z}_o \in S$ such that $G(\bar{z}_o) < d$, then

$$(3.3-1) \qquad \lim_{n \to \infty} u^{(n)} = \hat{u}$$

$$(3.3-2) \qquad \lim_{n \to \infty} H(z^{(n)}) = \operatorname*{Max}_{\substack{G(z) \leq d \\ z \in S}} H(z)$$

$$(3.3-3) \qquad H(\hat{z}) - \hat{u} \cdot [G(\hat{z}) - d] = \operatorname*{Max}_{z \in S} \{H(z) - \hat{u} \cdot [G(z) - d]\}$$

Proof:

Since the algorithm finds two extreme points of the set Y^o at each iteration of the decomposition algorithm for concave programs, it must converge by Theorem 18.

3.4 NONCONVEX CONSIDERATIONS

In the previous sections of this chapter the emphasis has been on relating the dual algorithm presented here to Dantzig's decomposition algorithm for concave programs. However, by Theorems 3 and 4 of Section (2.1), we have that $L(u)$ is a convex continuous function of u for all $u \geq 0$; and this result does not explicitly depend on the nature of the set S, or on the properties of the functionals $H(z)$ and $G(z)$ defined on S . Hence the dual problem

$$(3.4-1) \qquad\qquad \text{Inf } L(u) \; , \\ u \geq 0$$

where $L(u)$ is given by

$$(3.4-2) \qquad L(u) = \text{Sup}_{z \epsilon S} \{ H(z) - u \cdot [G(z) - d] \} ,$$

may be solved in place of the primal problem even when the primal is not a concave program. Since evaluating $L(u)$ defines a linear supporting function of $P(y)$, assuming the graph of $P(y)$ is closed, if it can be shown that the solution of the primal problem is on the convex hull of

$$(3.4-3) \qquad Y^o = \{y_o, y \mid y_o \leq H(z), y \geq G(z), z \epsilon S\} ,$$

then, noting Theorem 12 of Section (2.3), the dual algorithm may be used to produce this solution.

For that particular nonconvex problem that satisfies the conditions of Theorem 11, Section (2.3), we have the following much stronger result.

Theorem 22:

Let S be a convex compact space, $H(z) - u \cdot [G(z) - d]$ a
functional defined on $S \times \{u \mid u \geq 0\}$, quasi-concave and
upper semi-continuous in z for all $u \geq 0$, and let $\{u^{(n)}\}$
be the sequence of multipliers generated by Algorithm 2,
Section (3.2). Then

(3.4-3)
$$\lim_{n \to \infty} u^{(n)} = \hat{u}$$

(3.4-4)
$$\lim_{n \to \infty} L(u^{(n)}) = \inf_{u \geq 0} L(\hat{u})$$

Further, if there exists $\bar{z}_o \in S$ such that $G(\bar{z}_o) < d$,
then the Inf is always attained and may be replaced by Min.

CH _ ᵀER 4

DYNAMIC ECONOMIC PLANNING

4.1 THE OPTIMAL ADJUSTMENT OF THE CAPACITY OF A FIRM

As indicated in the introduction, an economic problem gave rise to the
more general results presented in this paper. Therefore, in this chapter
we formulate an economic example for which the theory is applicable. The
particular problem was chosen because it has been considered by others, is
relatively straight-forward to explain, and emphasizes the *dynamic* nature
of the policies of a firm. It was the desire to actually compute these
dynamic policies for the firm under nontrivial assumptions that motivated
the theory presented.

We will consider a multi-divisional firm where each division produces
a separate commodity. Further, we will assume that the production structure
is separable in the sense that the policies of one division only effect
the policies of another through demands for the same resources [7]. If
none of the firm's resources are binding then the policies of the different
divisions are completely independent. For simplicity, we assume that the
production structure of each division can be represented by a homothetic
production function with an analytic form, and that storage is not allowed.
(See Appendix B for definitions and examples of homothetic production
functions.) In general the problem to be considered is, given the demand
for each commodity as a function of time, determine the capacity of a firm
to meet this demand as a function of time subject to budgetary and other
resource constraints.

Let $\xi^k(t)$, defined for $t \in [0, T]$, be the demand at time t for the

k^{th} commodity of the firm; and assume $\xi^k(t)$ is continuous and has a finite number of relative maxima in the interval $[0, T]$. Let $y^k(t)$, defined for $t \in [0, T]$, represent the capacity of the firm at time t to produce the kth commodity, and let $v^k(t)$, defined for $t \in [0, T]$, be the actual level of production at time t of the kth commodity. Since storage is not allowed for this simple model, we have

$$(4.1-1) \qquad v^k(t) \leq \text{Min} \; (y^k(t) \; , \; \xi^k(t)) \quad k \in K \; .$$

Letting $F^k(\phi^k(x))$ be the homothetic production function for the kth commodity, a nonnegative vector $x^k(t)$ of inputs at time t can produce a nonnegative amount $v^k(t)$ of the kth commodity at time t if and only if

$$(4.1-2) \qquad F^k(\phi^k(x^k(t))) \geq v^k(t) \quad k \in K \; .$$

Assuming the supply of inputs is not restrictive, the net production income for the kth division at time t (sales revenue less production costs) is given by

$$(4.1-3) \qquad r^k \; v^k(t) - p^k(p) \; f^k(v^k(t))^{\dagger} \quad k \in K$$

where the selling price of the kth commodity r^k and the vector of purchase prices of the inputs p are assumed constant. Further, assume that we can increase capacity at a positive cost proportional to the rate of increase, i.e.,

$$(4.1-4) \qquad c^k \; \dot{y}^k(t) \quad \text{if} \quad \dot{y}^k(t) \geq 0 \; ,$$

---

†See Appendix B for this representation of a homothetic cost function and note that $F^{-1}(\cdot) \triangleq f(\cdot)$.

and decrease capacity at a salvage value proportional to the rate of decrease but less than or equal to the cost of an equivalent increase, i.e.,

$$(4.1-5) \qquad \gamma^k \, c^k \, \dot{y}^k(t) \quad \text{if} \quad \dot{y}^k(t) < 0 \quad \text{and} \quad 0 \leq \gamma^k \leq 1 \,.$$

Hence the present value of the income of the firm for all commodities over the time interval $[0, T]$ is given by

$$(4.1-6) \qquad \sum_{k \in K} \int_0^T \left\{ r^k \, v^k(t) - P^k(p) \, f^k(v^k(t)) \right.$$

$$\left. - c^k \, \dot{y}^k(t) \left[\delta(\dot{y}^k(t)) + \gamma^k (1 - \delta(\dot{y}^k(t))) \right] \right\} \, e^{-\alpha t} \, dt$$

$$(4.1-7) \qquad \text{where} \quad \delta(z) = \begin{cases} 1 & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}$$

and α is the interest rate.

The problem is then to choose the capacity function $y(t)$, with the initial capacity $y(0)$ and the time horizon T known, that maximizes (4.1-6) subject to (4.1-1) and some typical constraints. For example,

$$(4.1-8) \qquad L^k \leq \dot{y}^k(t) \leq M^k \quad t \in [0,T] \qquad k \in K$$

$$(4.1-9) \qquad \sum_{k \in K} \int_0^T \left\{ c^k \, \dot{y}^k(t) \left[\delta(\dot{y}^k(t)) + \gamma^k (1 - \delta(\dot{y}^k(t))) \right] \right\} \, e^{-\alpha t} \, dt \leq B \,.$$

Relation (4.1-8) bounds the rate at which we may change the capacity and (4.1-9) bounds the expenditures on these changes in capacity by a given budget.

If we now let $z(t) \equiv \dot{y}(t)$ so that

$$(4.1\text{-}10) \qquad y(t) = y(0) + \int_0^t z(s)\, ds$$

we can summarize the problem formulated above as follows:

Example 1

$$(4.1\text{-}11) \quad \text{Max} \sum_{k \in K} \int_0^T \left\{ r^k\, v^k(t) - P^k(p)\, f^k(v^k(t)) - \right.$$
$$\left. - c^k\, z^k(t)\left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t} dt$$

subject to

$$s_1^k = \left\{ z^k, v^k \;\middle|\; \begin{array}{ll} L^k \le z^k(t) \le M^k & t \in [0,\, T] \\[2mm] 0 \le v^k(t) \le \text{Min}\left(y^k(0) + \int_0^t z^k(s)ds,\; \xi^k(t) \right) \end{array} \right\} \; k \in K$$

$$\sum_{k \in K} \int_0^T \left\{ c^k\, z^k(t)\left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t}\, dt \le B\; .$$

The above example only has one global constraint, the budget constraint. This convenient formulation depends upon the assumption that the available inputs are not restrictive and thus the minimum cost function at time t for the kth commodity does not depend explicitly on $x^k(t)$.

We will now complicate Example 1 by allowing the input space to be bounded. We add constraints for those factors of production that are limited in supply. Let $I = \{1, 2, \ldots, I\}$ be the index set of these factors.

$$(4.1\text{-}12) \qquad \sum_{k \in K} \int_0^T a_i^k\, x_i^k(t)\; dt \le x_i^0 \quad i \in I$$

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$$(4.1-13) \quad \text{where } a_i^k = \begin{cases} +1 & \text{if the } i^{th} \text{ resource is consumed by the } k^{th} \text{ process} \\ -1 & \text{if the } i^{th} \text{ resource is produced by the } k^{th} \text{ process} \\ 0 & \text{otherwise.} \end{cases}$$

Defining a_i^k in this manner allows for producing some commodities that are inputs in the production of other commodities. Thus we can summarize the extended problem as follows:

Example 2

$$(4.1-14) \quad \text{Max} \sum_{k \in K} \int_0^T \left\{ r^k v^k(t) - p \cdot x^k(t) - c^k z^k(t) \left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t} dt$$

subject to

$$s_2^k = \left\{ z^k, x^k, v^k \left| \begin{array}{ll} L^k \leq z^k(t) \leq M^k & t \in [0, T] \\ 0 \leq x^k(t) & \\ 0 \leq v^k(t) \leq F^k(\phi^k(x^k(t))) & \\ v^k(t) \leq \text{Min} \left(y^k(0) + \int_0^T z^k(r) \, ds, \xi^k(t) \right) \end{array} \right. \right\} \quad k \in K$$

$$\sum_{k \in K} \int_0^T a_i^k x_i^k(t) \, dt \leq x_i^0 \qquad i \in I$$

$$\sum_{k \in K} \int_0^T \left\{ c^k z^k(t) \left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t} dt \leq B.$$

4.2 APPLICABILITY OF THE THEORY

In order to demonstrate that the theory is applicable, we need only verify that the conditions of the theorems of Chapter 3 are satisfied. We will assume that the strategies under consideration in both examples are chosen from the space L_2 consisting of numerical functions g defined on $[0, T]$ (to within a set of measure zero) with norm defined by

$$(4.2-1) \qquad ||g|| = \left[\int_0^T |g(t)|^2 dt \right]^{\frac{1}{2}} < + \infty .$$

It is well known that L_2 is a complete normed linear space and thus a Banach space.[†] Since L_2 is also reflexive, we can use the weak* topology on L_2, under which the strategy sets will be compact.

We will first consider Example 1. The strategy sets $S_1^k, k \in K$ are convex by the following lemma.

Lemma 23:

The set

$$S_1 = \left\{ z, v \ \middle| \ \begin{array}{l} L \leq z(t) \leq M \qquad\qquad\qquad t \in [0, T] \\[2mm] 0 \leq v(t) \leq \mathrm{Min}\left(y(0) + \int_0^t z(s)ds, \ \xi'(s) \right) \end{array} \right\}$$

is convex in (z, v) .

Proof:

The first constraint is linear in z and the second is linear in v. Therefore, we need only show that

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[†] See Berge [6] p. 252.

$$\text{Min} \left(y(0) + \int_0^t z(s)ds, \ \xi(t) \right)$$

is a concave function in z and hence

$$v(t) - \text{Min} \left(y(0) + \int_0^t z(s)ds, \ \xi(t) \right)$$

is a convex function. We have that $y(0) + \int_0^t z(s)ds$ is linear in

z and $\xi(t)$ is constant in z. Thus $\text{Min} \left(y(0) + \int_0^t z(s)ds, \ \xi(t) \right)$

is a concave function in z, since the minimum of two concave functions

is concave.

To establish that the global constraint is a convex function we prove the

following lemma.
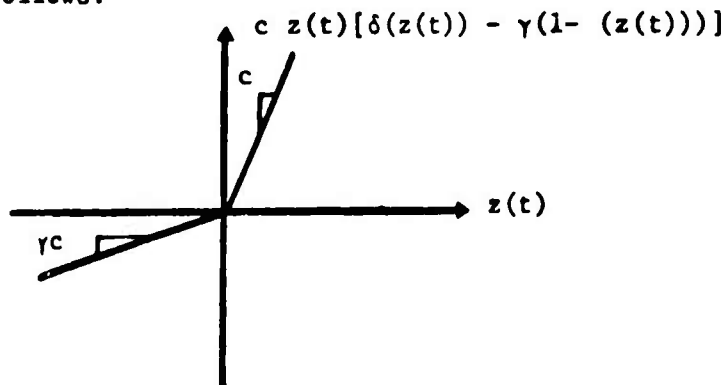
Lemma 24:

If $0 \leq \gamma \leq 1$, then the capacity adjustment cost function

$$\int_0^T \{c \ z(t)[\delta(z(t)) - \gamma(1-\delta(z(t)))]\} \ e^{-\alpha t} \ dt \quad \text{is convex in } z.$$

Proof:

The result is easily seen by graphing the indicated function for

fixed t as follows:

Since $e^{-\alpha t} \geq 0 \ \forall \ t$, the integral of interest is convex in z.

The objective function of Example 1 is concave in (z, v) since the total income is the sum of the net production income

$$(4.2\text{-}2) \qquad \sum_{k \in K} \int_0^T \left| r^k \, v^k(t) - P^k(p) \, f^k(v^k(t)) \right| \, e^{-\alpha t} \, dt \ ,$$

which is concave in v^k if $f^k(v^k)$ is convex in v^k, and the negative of the capacity adjustment cost function

$$(4.2\text{-}3) \qquad - \sum_{k \in K} \int_{0.}^T \left| c^k \, z^k(t) [\delta(z^k(t)) + \gamma^k(1-\delta(z^k(t)))] \right| e^{-\alpha t} \, dt$$

which is concave in z^k by Lemma 15.

In Example 2, we have strategy sets S_2^k $k \in K$ that differ from those of Example 1 only by the nonnegativity restriction on $x^k(t)$ and the production constraint which is convex by the following lemma.

Lemma 25:

If $f(v)$ is a convex increasing function in v and $\phi(x)$ is a concave function in x, then

$$v(t) - F(\phi(x(t)))$$

is a convex function in (v, x) .

Proof:

Since $f(\cdot)$ is the inverse of $F(\cdot)$ by definition and $f(\cdot)$ is a convex increasing function, $F(\cdot)$ is a concave increasing function.

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A concave increasing transformation of a concave function is
concave,[†] and thus $F(\phi(x(t)))$ is concave in x and
$v(t) - F(\phi(x(t)))$ is convex in (x,v) .

Hence the strategy sets S_2^k are convex in (z^k, x^k, v^k) , $k \in K$. The
additional global constraints in Example 2 on the factors of production that
are limited (4.1-12) are convex since they are linear in x^k . Finally, the
objective function is concave since it is linear in x^k and v^k , and
concave in z^k by Lemma 24.

The last condition to verify for the theorems of Chapter 3 is that the
maxima of the Lagrangian subproblems is taken on for all $u \geq 0$. However,
as this depends on the solution technique employed, we will consider this
point in the next section. First, we will show the subproblems for Example 1
and Example 2 can be considered to have the same structure. The Lagrangian
subproblems for Example 1 are

$$(4.2-4) \qquad \text{Max} \int_0^T \left\{ r^k v^k(t) - P^k(p) f^k(v^k(t)) - \right.$$
$$\left. - (c^k + \bar{u}_B) z^k(t) \left[\delta(z^k(t) + \gamma^k(1-\delta(z^k(t)))) \right] \right\} e^{-\alpha t} dt$$

subject to

$$S_1^k = \left\{ z^k, v^k \;\middle|\; \begin{array}{ll} L^k \leq z^k(t) \leq M^k & t \in [0, T] \\ 0 \leq v^k(t) \leq \text{Min} \left(y^k(0) + \int_0^t z^k(s)ds, \; \xi^k(t) \right) \end{array} \right\}$$

where $\bar{u}_B$ is the current Lagrange multiplier associated with the global

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[†]See Berge [6], Page 191.

budget constraint. The Lagrangian subproblems for Example 2 are

$$
(4.2\text{-}5) \quad \text{Max} \int_0^T \left\{ r^k v^k(t) - \dot{p}^k \cdot x^k(t) - \right.
$$
$$
\left. - (c + \tilde{u}_B) z^k(t) \left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t} \, dt
$$

subject to

$$
S_2^k = \left\{ z^k, x^k, v^k \; \middle| \;
\begin{array}{ll}
L^k \leq z^k(t) \leq M^k & t \in [0, T] \\[2mm]
0 \leq x^k(t) & \\[2mm]
0 \leq v^k(t) \leq F^k(\phi^k(x^k(t))) & \\[2mm]
v^k(t) \leq \text{Min} \left(y^k(0) + \int_0^t z^k(s)ds, \; \xi^k(t) \right) &
\end{array}
\right\}
$$

$$
\text{where } \tilde{p}_i^k = p_i + a_i^k \tilde{u}_i \quad i \in I
$$

and $\tilde{u} \in E^J$ is the vector of Lagrange multipliers associated with the
constraints on the outputs. If we have formulated the problem correctly, the
optimal solution will be finite; and thus the optimal Lagrange multipliers
will be such that the associated $\tilde{p}^k \geq 0$. Although $x^k(t)$ is restricted
in Example 2, it is only restricted to nonnegativity in the associated
Lagrangian subproblems. Hence we may use the cost function representation
to reduce the Lagrangian subproblems for Example 2 to

$$
(4.2\text{-}6) \quad \text{Max} \int_0^T \left\{ r^k v^k(t) - P^k(\bar{p}) \, \varepsilon^k(v^k(t)) - \right.
$$
$$
\left. - (c + \tilde{u}_B) z^k(t) \left[\delta(z^k(t)) + \gamma^k(1 - \delta(z^k(t))) \right] \right\} e^{-\alpha t} \, dt
$$

subject to

$$S_2^k = \left\{ z^k, v^k \ \middle| \ \begin{array}{ll} L^k \leq z^k(t) \leq M^k & t \in [0, T] \\[2mm] 0 \leq v^k(t) \leq \text{Min} \left(y^k(0) + \displaystyle\int_0^t z^k(s)ds, \ \xi(t) \right) \end{array} \right\}$$

where $\quad \tilde{p}_i^k = p_i + a_i^k \tilde{u}_i \qquad i \in I$.

Note that (4.2-6) and (4.2-4) have the same form.

4.3 SOLUTION OF THE SUBPROBLEMS

In order to emphasize the alternatives presented, we will note two solution techniques for the Lagrangian subproblems - one exact and one approximate. For the exact technique, the strategy set will be restricted to strategies of expansion only, i.e.,

$$(4.3-1) \qquad 0 = L^k \leq z^k(t) \leq M^k \quad \forall t \in [0, T] \qquad k \in K$$

and the homothetic production structure will be restricted to constant returns to scale,[†] i.e.,

$$(4.3-2) \qquad \qquad f(v) = v .$$

Noting that $\tilde{p}_1^k = p_1 + a_1^k \tilde{u}_1$ for Example 2 and letting $\tilde{p}_1^k = p_1$ for Example 1, the Lagrangian subproblems for both examples may then be written

$$(4.3-3) \qquad \text{Max} \int_0^T \left\{ (r^k - P^k(\tilde{p}^k)) v^k(t) - (c + \tilde{u}_B) z^k(t) \right\} e^{-\alpha t} dt$$

subject to

$$S^k = \left\{ z^k, v^k \; \middle| \; \begin{array}{ll} 0 \leq z^k(t) \leq M^k & t \in [0, T] \\[2ex] 0 \leq v^k(t) \leq \text{Min}\left(y^k(0) + \int_0^t z^k(s), \; \xi^k(t) \right) \end{array} \right\}$$

if $r^k \leq p^k(\tilde{p}^k)$, then the optimal solution to the kth subproblem is $z^k(t) = 0 \quad \forall t \in [0, T]$. If $r^k > P^k(\tilde{p}^k)$, then the optimal solution to the kth subproblem will be such that

---

[†]See Lemma D of Appendix B.

$$(4.3\text{-}4) \qquad v^k(t) = \text{Min}\left(y^k(0) + \int_0^t z^k(s), \; \xi^k(t)\right) \qquad t \; \varepsilon \; [0, T]$$

and (4.3-3) reduces to

$$(4.3\text{-}5) \qquad \text{Max} \int_0^T \left\{ (r^k - p^k(\tilde{p}^k))\text{Min}\left(y^k(0) + \int_0^t z^k(s)ds, \; \xi^k(t)\right) - \right.$$
$$\left. - (c^k + \tilde{u}_B) \; z^k(t) \right\} e^{-\alpha t} \; dt$$

subject to

$$S^k = \{z^k \mid 0 \leq z^k(t) \leq M^k, \; t \; \varepsilon \; [0, T]\} \; .$$

An exact solution technique for precisely this problem (4.3-5) has been given by Arrow, Beckmann, and Karlin [1]. Their method employs a minimax theorem and considers the following problem, which is equivalent to (4.3-5).

$$\underset{\substack{z^k \varepsilon S^k \; \alpha^k \varepsilon A^k}}{\text{Max} \quad \text{Min}} \int_0^T \left\{ (1 - \alpha(t))(r^k - p^k(\tilde{p}^k))\xi(t) \right.$$
$$+ \alpha(t)(r^k - p^k(\tilde{p}^k))\left(y^k(0) + \int_0^t z^k(s)ds\right)$$
$$(4.3\text{-}6) \qquad \left. -(c^k + \tilde{u}_B) \; z^k(t) \right\} e^{-\alpha t} \; dt$$

where

$$S^k = \{z^k \mid 0 \leq z^k(t) \leq M^k, \; t \; \varepsilon \; [0, T]\}$$
$$A^k = \{\alpha^k \mid 0 \leq \alpha^k(t) \leq 1, \; t \; \varepsilon \; [0, T]\} \; .$$

The strategy sets S^k and A^k are compact in the weak* topology by Alaoglu's Theorem,[†] and since the functional $H^k(z^k, \alpha^k)$ defined on $S^k \times A^k$ is linear in each variable separately, it is continuous in each in the weak*

[†]See Berge [6] p. 262

topology. Hence the indicated Max in (4.3-5) exists. Therefore, the problem of optimal capacity expansion of a multi-commodity firm with constant returns to scale can then be handled by an algorithm proposed in this paper with the technique of Arrow, Beckmann, and Karlin employed to solve the subproblems at each iteration. In Example 2, the solution of the Lagrangian subproblems must be translated into the required form if a primal algorithm is to be employed. This is easily accomplished, since a vector of inputs $x^k(t)$ yielding minimum cost at time t is given by

$$(4.3-7) \qquad x^k(t) = f^k(v^k(t)) \frac{\bar{x}^k}{f^k(1)} \qquad t \in [0, T]$$

where $\qquad \bar{x}^k$ is determined by[†]

$$p^k \bar{x}^k = \text{Min } p^k \cdot x \qquad k \in K .$$

$$(4.3-8) \qquad \qquad \text{Subject to}$$

$$F^k(\phi^k(x)) \geq 1 .$$

We will now consider the approximate technique of dynamic programming for the solution of the Lagrangian subproblems of the form (4.2-6). The continuous demand functions $\xi^k(t)$ are then replaced by discrete approximations, with intervals normalized to length one, and numbered in the reverse order, such that

$$(4.3-9) \qquad \xi_n^k = \frac{\int_{t_{n+1}}^{t_n} \xi^k(t)dt}{t_n - t_{n+1}} = \int_{t_{n+1}}^{t_n} \xi^k(t)dt \qquad k \in K .$$

[†] See Lemma E of Appendix B .

A typical approximation is given below where the number of intervals is chosen by the degree of accuracy desired.
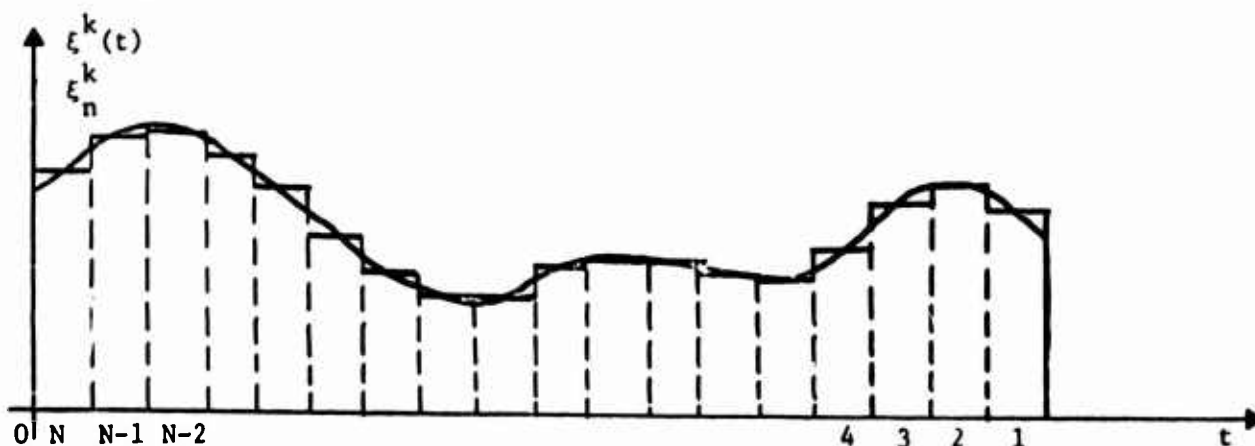
If we define y_n^k as the capacity for the kth commodity in the nth interval, we have

(4.3-10)
$$y_n = y_{n+1} + z_{n+1}$$

where z_n is the adjustment that takes place in the nth interval. Letting v_n^k be the level of production of the kth commodity in the nth interval and β be the discount factor associated with interest rate α , the approximation to the kth Lagrangian subproblem is

$$\text{Max} \sum_{n=1}^{N} \left\{ r^k v_n^k - P^k(\bar{p}^k) \ f^k(v_n^k) - \right.$$

$$\left. - (c^k + \bar{u}_B) \ z_n^k (\delta(z_n^k) + \gamma^k(1-\delta(z_n^k))) \right\} \ \beta^n$$

(4.3-11) subject to

$$S^k = \left\{ z^k, v^k \ \middle| \ \begin{matrix} L^k \leq z_n^k \leq M^k \\ \\ 0 \leq v_n^k \leq \text{Min}\left(y_n^k, \ \xi_n^k \right) \end{matrix} \right\}$$

where $\bar{p}_i = p_i + a_i^k \bar{u}_i$ $\qquad$ $i \in I$.

Since

(4.3-12)

$$L^k \leq z_n^k \leq M^k$$

$$0 \leq v_n^k \leq \max_{t \in [0,T]} \xi(t) < +\infty \ ,$$

S^k is a closed bounded set in euclidian n-space; and hence S^k is compact.
Therefore, the indicated Max in (4.3-11) exists. The recursive
relationship used to solve the Lagrangian subproblems is then

$$\pi_n^k\left(y_n^k\right) = \max \left\{ r^k v_n^k - \Gamma^k(\tilde{p}^k) f^k\left(v_n^k\right) \right.$$

$$- (c + u_B) z_n^k\left(\delta\left(z_n^k\right) + \gamma^k\left(1-\delta\left(z_n^k\right)\right)\right)$$

$$\left. + \beta^{n-1} \pi_{n-1}^k\left(y_n^k + z_n^k\right)\right\}$$

(4.3-13)   subject to

$$L^k \leq z_n^k \leq M^k$$

$$0 \leq v_n^k \leq \min\left(y_n^k, \ \xi_n^k\right)$$

where

$$\pi_0^k\left(y_0^k\right) \triangleq 0 \ .$$

This is a straight forward one dimensional problem that is relatively easy
to solve on a computer. Hence, both examples may be handled by the algorithms
proposed, with dynamic programming used to approximate the solutions of the
Lagrangian subproblems.

APPENDIX A

MINIMAX THEOREMS

We state here the two powerful minimax theorems employed in the derivation of the duality theory; one due to Sion[26] and the other to Fan [14].

Lemma A: *Sion's Minimax Theorem*

Let M and N be convex topological spaces, one of which is compact, and $F(u,v)$ a functional defined on $M \times N$, quasi-concave and upper semi-continuous in u for all $v \in N$, quasi-convex and lower semi-continuous in v for all $u \in M$. Then

$$\underset{u \in M}{\text{Sup}} \, \underset{v \in N}{\text{Inf}} \, F(u,v) = \underset{v \in N}{\text{Inf}} \, \underset{u \in M}{\text{Sup}} \, F(u,v) \, .$$

Lemma B: *Fan's Minimax Theorem*

Let M be a convex set and N a convex compact Hausdorf space. Let $F(u,v)$ be a functional defined on $M \times N$, concave in u for all $v \in N$, convex and lower semi-continuous in v for all $u \in M$. Then

$$\underset{u \in M}{\text{Sup}} \, \underset{v \in N}{\text{Inf}} \, F(u,v) = \underset{v \in N}{\text{Inf}} \, \underset{u \in M}{\text{Sup}} \, F(u,v) \, .$$

APPENDIX B

HOMOTHETIC COST AND PRODUCTION FUNCTIONS

Homothetic cost and production functions were first defined by Shephard
[24] and are extensively treated in his recent work [25]. In this appendix,
we define the functions, give some examples, and note the properties that
are employed elsewhere.

Definition:[†]

A homothetic production function is one of the form $F(\phi(x))$ where
$\phi(x)$ is nonnegative, homogeneous of degree one, nondecreasing in x,
upper semi-continuous and quasi-concave for all $x \in D = \{x \mid x_i \geq 0 \ \forall i\}$;
and $F(\cdot)$ is nonnegative, continuous and strictly increasing with
$F(0) = 0$.

In what follows the inverse function of $F(\cdot)$, which always exists since
$F(\cdot)$ is strictly increasing, will be denoted by $f(\cdot)$. The following
lemma gives the most important property of homothetic production functions.

Lemma C:

The isoquant for any output rate $u > 0$ of a homothetic production
function may be obtained from that for unit output rate by radical
expansion from the origin in a fixed ratio $f(u)/f(1)$.

The class of homothetic production functions includes all of the production
functions commonly employed. Some examples are as follows:

[†]All definitions and lemmas are taken from Shephard [25]

Cobb-Douglas

$$\phi(x) = A \prod_{i=1}^{r} x_i^{a_i} \quad A > 0, \ a_i > 0, \ \sum_{i=1}^{n} a_i = r$$

B-1

$$f(u) = u^{1/r}$$

Arrow-Chenery-Minhas-Solow (ACMS)

$$\phi(x) = \left[\sum_{i=1}^{n} a_i x_i^{-b} \right]^{-\frac{1}{b}} \quad a_i > 0 \quad b > -1$$

B-2

$$f(u) = u$$

Uzawa (Generalized ACMS)

$$\phi(x) = \prod_{j=1}^{J} \left[\sum_{i \in N_j} a_i x_i^{-b_j} \right]^{-\frac{\rho_i}{b_j}}$$

B-3

$$a_i > 0, \ b_j > -1, \ \rho_j > 0, \ \sum_{j=1}^{s} \rho_j = 1$$

$$f(u) = u$$

The function $f(\cdot)$ completely discribes the returns to scale as follows.

Lemma D:

$f(u)$ convex implies nonincreasing returns to scale, $f(u)$ concave implies nondecreasing returns to scale, and $f(u)$ linear implies constant returns to scale.

For any price direction $p \in D = \{p | p_i \geq 0 \ \forall i\}$ and nonnegative level of output $u \geq 0$, there is associated with any production function a cost function given by

$$Q(u, p) = \text{Min } p \cdot x$$

B-4
$$F(\phi(x)) \geq u$$

$$x \in D \quad .$$

Note that the cost function is defined for $x \geq 0$ but otherwise unrestricted. For a homothetic production function the cost function has a more specific form given by

B-5
$$Q(u, p) = P(p) \, f(u) \quad .$$

This convenient representation for the cost function and the radial expansion property of Lemma G yield the following useful result.

Lemma E:

If the vector $\bar{x}(1)$ produces one unit of output at minimum cost then the vector $\bar{x}(u) = \frac{f(u)}{f(1)} \bar{x}(1)$ produces u units of output at minimum cost.

Proof:

By definition $Q(u, p) = \underset{F(\phi(x)) \geq u}{\text{Min }} p \cdot x = p \cdot \bar{x}(u)$ and for a homothetic cost

function $Q(u, p) = P(p) \, f(u)$. Therefore,

$$P(p) = p \cdot \frac{\bar{x}(u)}{f(u)} = p \cdot \frac{\bar{x}(1)}{f(1)}$$

and hence $p \cdot \bar{x}(u) = p \cdot \frac{f(u)}{f(1)} \bar{x}(1)$. Therefore, the input vector $\frac{f(u)}{f(1)} \bar{x}(1)$ produces u units of output at minimum cost.

Corresponding to each of the examples of homothetic production functions, we have the following homothetic cost functions:

74

Cobb-Douglas

B-6

$$Q(u, p) = \left[\frac{1}{A} \prod_{i=1}^{n} \left(\frac{a_i}{p_i}\right)^{-a_i}\right] u^{1/r}$$

$$A > 0 \quad a_i > 0 \quad \sum_{i=1}^{n} a_i = r$$

Arrow-Chenery-Minhas-Solow (ACMS)

B-7

$$Q(u, p) = \left[\sum_{i=1}^{n} a_i \left(\frac{p_i}{a_r}\right)^{\frac{b}{b+1}}\right]^{\frac{b+1}{b}} u$$

$$a_i > 0 \quad b > -1$$

Uzawa (Generalized ACMS)

B-8

$$Q(u, p) = \prod_{j=1}^{s} \left(\frac{1}{\rho_j}\right)^{\rho_j} \left[\sum_{i \in N_j} a_i \left(\frac{p_i}{a_i}\right)^{\frac{b_i}{b_j+1}}\right]^{\rho_j \left(\frac{b_i+1}{b_j}\right)} u$$

$$a_i > 0, \quad b_j > -1, \quad \rho_j > 0, \quad \sum_{j=1}^{s} \rho_j = 1 \ .$$

REFERENCES

[1] Arrow, K. J., M. J. Beckmann, and S. Karlin, "The Optimal Expansion of the Capacity of a Firm," from STUDIES IN THE MATHEMATICAL THEORY OF INVENTORY AND PRODUCTION by Arrow, Karlin, and Scarf, Stanford University Press, Stanford, California, (1958).

[2] Arrow, K. J., and L. Hurwicz, "Reduction of Constrained Maxima to Saddle Point Problems," from PROCEEDINGS OF THIRD BERKELEY SYMPOSIUM ON MATHEMATICAL STATISTICS AND PROBABILITY, University of California Press, Berkeley, (1956).

[3] Balas, E., "An Infeasibility-Pricing Decomposition Method for Linear Programs," J. Oper. Res. Soc., Vol. 14, No. 5, (1966).

[4] Bellman, R., "Dynamic Programming and Lagrange-Multipliers," Proc. Nat. Acad. Sci. U.S.A., Vol. 42, (1956).

[5] Bellman, R., and S. Dreyfus, APPLIED DYNAMIC PROGRAMMING, Princeton University Press, Princeton, N.J., (1962).

[6] Berge, C., TOPOLOGICAL SPACES, Macmillan Book Company, Inc., New York, N.Y., (1963).

[7] Bradley, S. P., "General Production Structures and Their Related Functions," ORC 67-8, Operations Research Center, University of California, Berkeley.

[8] Brooks, R., and A. Geoffrion, "Finding Everett's Lagrange Multipliers by Linear Programming," J. Oper. Res. Soc., Vol. 14, No. 6, (1966).

[9] Dantzig, G. B., "Linear Control Processes and Mathematical Programming," J. SIAM Control, Vol. 4, No. 1, (1966).

[10] Dantzig, G. B., LINEAR PROGRAMMING AND EXTENSIONS, Princeton University Press, Princeton, N.J., (1963).

[11] Dantzig, G.B., and P. Wolfe, "The Decomposition Algorithm for Linear Programs," Econometrica, Vol. 29, No. 4, (1961).

[12] Everett, H., "Generalized Lagrange Multiplier Method for Solving Problems of Optimal Allocation of Resources," J. Oper. Res. Soc., Vol. 11, No. 3, (1963).

[13] Falk, J. E., "Theory of Lagrange Multipliers for Constrained Optimization Problems," (to appear).

[14] Fan, K., "Minimax Theorems," Proc. Nat. Acad. Sci., Vol. 39, No. 1, (1953).

[15] Gale, D., V. L. Klee, and R. T. Rockafellar, "Convex Functions on Convex Polytopes," Proc. Amer. Math. Soc., (to appear).

76

[16]   Huard, P., "Convex Programming--Dual Algorithm," ORC 63-21, Operations
         Research Center, University of California, Berkeley.

[17]   Hurwicz, L., "Programming in Linear Spaces," from STUDIES IN LINEAR
         AND NONLINEAR PROGRAMMING by Arrow, Hurwicz and Uzawah, Stanford
         University Press, Stanford, California, (1958).

[18]   Karamardian, S., "Duality in Mathematical Programming," ORC 66-2,
         Operations Research Center, University of California, Berkeley.

[19]   Kuhn, H. W., and A. W. Tucker, "Nonlinear Programming," from
         PROCEEDINGS OF THE SECOND BERKELEY SYMPOSIUM OF MATHEMATICAL
         STATISTICS AND PROBABILITY by Neyman, University of California
         Press, Berkeley, (1951).

[20]   Mangasarian, O. L., NONLINEAR PROGRAMMING, McGraw-Hill Book Company,
         New York, New York, (to appear).

[21]   Nunn, W. R., "The Lagrange Multiplier--A Heuristic Presentation,"
         Memorandum for the Director of Research, Operations Evaluation
         Group, (28 September 1965).

[22]   Ritter, K., "Duality for Nonlinear Programming in Banach Spaces,"
         SIAM J. Appl. Math., Vol. 15, No. 2, (1967).

[23]   Rockafellar, R. T., "Duality and Stability in Extremum Problems
         Involving Convex Functions," Pacific J. of Math., Vol. 21, No. 1,
         (1967).

[24]   Shephard, R. W., COST AND PRODUCTION FUNCTIONS, Princeton University
         Press, Princeton, New Jersey, (1952).

[25]   Shephard, R. W., COST AND PRODUCTION FUNCTIONS, Princeton University
         Press, Princeton, New Jersey, (to appear), Chapters available
         ORC 66-42, ORC 67-4, 5, 7, Operations Research Center,
         University of California, Berkeley.

[26]   Sion, M., "On General Minimax Theorems," Pacific J. of Math., Vol. 8,
         No. 1, (1958).

[27]   Takahashi, I., "Variable Separation Principle for Mathematical
         Programming," J. Oper. Res. Soc. of Japan, Vol. 6, No. 2, (1964).

[28]   Van Slyke, R., and R. Wets, "Programming Under Uncertainty and
         Stochastic Optimal Control," J. SIAM Control, Vol. 4, No. 1,
         (1966).

[29]   Wolfe, P., "A Duality Theorem for Nonlinear Programming,"
         Quart. J. Appl. Math., Vol. 19, No. 3, (1961).

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